3. BÖLÜM

ELEKTROSTATİKTE SINIR-DEĞER PROBLEMLERİ-II

3.1. Küresel koordinatlarda Laplace denklemi

$$\vec{\nabla}^2 \Phi(r,\theta,\varphi)=0$$

$$=\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)$$

$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

pot.'i tek değişkenli fonk.'ların çarpımı olarak düşünürsek,

$$\Phi(r,\theta,\varphi)=R(r)P(\theta)Q(\varphi)$$

$$\frac{PQ}{r}\frac{d^2}{dr^2}(rR)+\frac{RQ}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\frac{RP}{r^2\sin^2\theta}\frac{d^2Q}{d\varphi^2}=0$$

$$\frac{1}{rR}\frac{d^2}{dr^2}(rR)+\frac{1}{Pr^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\frac{1}{Qr^2\sin^2\theta}\frac{d^2Q}{d\varphi^2}=0$$

Her iki tarafı $r^2\sin^2\theta$ ile çarparak,

$$\underbrace{r^2\sin^2\theta\left\{\frac{1}{rR}\frac{d^2}{dr^2}(rR)+\frac{1}{Pr^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\right\}}_{+m^2}+\underbrace{\frac{1}{Q}\frac{d^2Q}{d\varphi^2}}_{-m^2}=0$$

$$\boxed{\frac{d^2 Q}{d\varphi^2} = -m^2 Q} \quad (\mathrm{I})$$

$$r^2 \left[\frac{1}{rR} \frac{d^2}{dr^2}(rR) + \frac{1}{P r^2 \sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) \right] = \frac{m^2}{\sin^2\theta}$$

$$rR = U \Rightarrow \underbrace{\frac{r^2}{U} \frac{d^2 U}{dr^2}}_{\ell(\ell+1)} = \underbrace{-\frac{1}{P\sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) + \frac{m^2}{\sin^2\theta}}_{\ell(\ell+1)}$$

$$\boxed{\frac{d^2 U}{dr^2} = \frac{\ell(\ell+1)}{r^2} U} \quad (\mathrm{II})$$

$$\frac{1}{P\sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right] = 0 \quad (\mathrm{III})$$

genelleştirilmiş Legendre dif. denk.'i

(I)'in çözümleri $Q(\varphi) = A e^{im\varphi} + B e^{-im\varphi}$

$\varphi : (0, 2\pi)$ aralında her değeri alabiliyor $\Rightarrow$ Q'nun tek değerli olabilmesi için $\underline{m}$'nin tamsayı olması gerekir, yani,

$$e^{\pm im\varphi} = e^{\pm im(\varphi + 2\pi)} \Leftrightarrow e^{\pm im 2\pi} = 1 \quad m \text{ tam sayı.}$$

$$m = 0, \pm 1, \pm 2, \ldots$$

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(II) $$\frac{d^2u}{dr^2} = \frac{\ell(\ell+1)}{r^2}u \qquad u = r\,R \text{ idi,}$$

$$u = A\,r^{\ell+1} + B\,r^{-\ell} \quad \text{sağlar, yani}$$

$$R = A\,r^{\ell} + B\,r^{-\ell-1}$$

III.)'ün özel hali $m=0$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \ell(\ell+1)P = 0$$

$x = \cos\theta$ dersek $\sin^2\theta = 1-x^2$ $\quad \frac{d}{dx} = \frac{d}{d(\cos\theta)} = -\frac{1}{\sin\theta}\frac{d}{d\theta}$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \ell(\ell+1)P = 0$$

$$P = \sum_{j=0}^{\infty} a_j\, x^{\alpha+j} \qquad P' = \sum_j a_j\,(\alpha+j)\,x^{\alpha+j-1}$$

$$x^2 P' = \sum_j a_j(\alpha+j)\,x^{\alpha+j+1}$$

$$(1-x^2)P' = \sum_j a_j\,(\alpha+j)\,x^{\alpha+j-1} - \sum_j a_j(\alpha+j)\,x^{\alpha+j+1}$$

$$\frac{d}{dx}\left[(1-x^2)P'\right] = \sum_j a_j\,(\alpha+j)(\alpha+j-1)\,x^{\alpha+j-2}$$

$$-\sum_j a_j(\alpha+j)(\alpha+j-1)\,x^{\alpha+j}$$

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$$\sum_j a_j(\alpha+j)(\alpha+j-1)x^{\alpha+j-2} - \sum_j a_j(\alpha+j)(\alpha+j+1)x^{\alpha+j}$$

$$+l(l+1)\sum_j a_j x^{\alpha+j} = 0$$

$j \to j+2$

$$a_0\alpha(\alpha-1)x^{\alpha-2} + a_1(\alpha+1)\alpha x^{\alpha-1} + \sum_{j=2} a_j(\alpha+j)(\alpha+j-1)x^{\alpha+j-2}$$

$$-\sum_{j=0} a_j(\alpha+j)(\alpha+j+1)x^{\alpha+j} + l(l+1)\sum_j a_j x^{\alpha+j} = 0$$

$$a_0\alpha(\alpha-1)x^{\alpha-2} + a_1(\alpha+1)\alpha x^{\alpha-1}$$

$$+\sum_{j=0} a_{j+2}(\alpha+j+2)(\alpha+j+1)x^{\alpha+j}$$

$$-\sum_{j=0} a_j(\alpha+j)(\alpha+j+1)x^{\alpha+j} + l(l+1)\sum_{j=0} a_j x^{\alpha+j} = 0$$

$$\to \sum_{j=0}\left\{ a_{j+2}(\alpha+j+2)(\alpha+j+1) - a_j\left[(\alpha+j)(\alpha+j+1) - l(l+1)\right]\right\}$$

$$\times x^{\alpha+j} = 0$$

$$+a_0\alpha(\alpha-1)x^{\alpha-2} + a_1(\alpha+1)\alpha x^{\alpha-1}$$

x katsayıların ayrı ayrı sıfır olması gerekir!

1) $a_0\alpha(\alpha-1)=0 \qquad a_0 \neq 0 \Rightarrow \alpha = 1$ ya da 0

2) $a_1\alpha(\alpha+1)=0 \qquad a_1 = 0$ olur.

$$a_{j+2} = a_j \frac{(\alpha+j)(\alpha+j+1) - l(l+1)}{(\alpha+j+1)(\alpha+j+2)} \qquad j=0,1,2,\dots$$

$\alpha=0 \Rightarrow P = a_0 + a_2 x^2 + a_4 x^4 + \dots$

$\alpha=1 \Rightarrow P = a_0 x + a_2 x^3 + a_4 x^5 + \dots$

(2) de $\begin{cases} a_1 \neq 0 \Rightarrow \alpha = 0, -1 \\ a_0 = 0 \end{cases}$ (I)

$\alpha=-1 \Rightarrow P = a_1 + a_3 x^2 + a_5 x^4 + \dots$

$\alpha=0 \Rightarrow P = a_1 x + a_3 x^3 + \dots$

ikisi de aynı şey!

$x^2 < 1$ için (I) toplamları yakınsaktır.

$-1 \leq x \leq +1$ $\quad x = \pm 1$ için (I)'ler sonsuz verecek. Toplamın sonlu olmasını istiyorum. Olması için l'nin tamsayı olması gerekir.

l: çift tamsayı $\Rightarrow \alpha = 0$ olmalı $j = l$ 'de kesilir,

$\alpha = 1$ kesilmez.

$P_l = a_0 + a_2 x^2 + \dots$

l: tek tam sayı $\Rightarrow \alpha = 1$ olmalı $j = l-1$ de kesilir.

$$P_\ell = a_0 x + a_2 x^3 + \dots$$

$$P_\ell = \begin{cases} a_0 + a_2 x^2 + \dots + a_\ell x^\ell \Leftarrow \ell \text{ çift} \\ a_0 x + a_2 x^3 + \dots + a_{\ell-1} x^{\ell-1} \Leftarrow \ell \text{ tek} \end{cases}$$

$$P_0 = a_0$$

$$P_1 = a_0 x$$

$$P_2 = a_0 + a_2 x^2$$

$$P_3 = a_0 x + a_2 x^3$$

$$P_4 = a_0 + a_2 x^2 + a_4 x^4$$

$j=0 \Rightarrow a_2$'yi bulalım, bu çift olduğundan $\alpha = 0$

$$a_2 = \frac{-2\times 3}{1\times 2} a_0 = -3a_0 \qquad \ell = 2 \qquad P_2(x) = \frac{a_0}{2}(1-3x^2)$$

$j=2 \Rightarrow a_4 \quad (\alpha = 0) \qquad \ell = 4$

$$a_4 = \frac{2\times 3 - 4\times 5}{3\times 4} a_2 = -\frac{7}{6} a_2 = -\frac{21}{6} a_0 = \frac{7}{2} a_0$$

$$P_4(x) = a_0 (1 - 3x^2 + \frac{7}{2} x^4)$$

$j=0 \Rightarrow a_2$'yi bulalım $\ell = 3 \quad (\alpha = 1)$

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$$a_2 = \frac{1\times 2 - 3\times 4}{2\times 3}a_0 = -\frac{10}{6}a_0 = -\frac{5}{3}a_0$$

$$P_3(x) = a_0\left(x - \frac{5}{3}x^3\right)$$

Bağlamın 1'e normalize etmek mümkün

$P_\ell(x=+1)=1$ olsun diyelim

$$P_0(x)=1$$

$$P_1(x)=x$$

$$P_2(x)=\frac{-1}{2}(1-3x^2)=\frac{1}{2}(3x^2-1)$$

$$P_3(x)=\frac{-3}{2}\left(x-\frac{5}{3}x^3\right)=\frac{1}{2}(5x^3-3x)$$

$$P_\ell(x)=\frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$$ Rodrigues toplu gösterimi

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_\ell}{dx}\right]+\ell(\ell+1)P_\ell(x)=0$$

$$\int_{-1}^{+1} dx\,\underbrace{P_{\ell'}}_{u}\,\underbrace{\frac{d}{dx}\left[(1-x^2)\frac{dP_\ell}{dx}\right]}_{dv}+\int_{-1}^{+1}\ell(\ell+1)P_\ell P_{\ell'}\,dx=0$$

$$\int_{-1}^{+1} dx\,\underbrace{P_\ell}_{u}\,\underbrace{\frac{d}{dx}\left[(1-x^2)\frac{dP_{\ell'}}{dx}\right]}_{dv}+\int_{-1}^{+1}\ell'(\ell'+1)P_{\ell'}P_\ell\,dx=0$$

$$-\int_{-1}^{+1} dx\, P_{\ell'} \left[(1-x^2) P_\ell'\right]' + \ell(\ell+1) \int_{-1}^{+1} (\quad) = 0$$

$$-\int_{-1}^{+1} dx\, P_\ell \left[(1-x^2) P_{\ell'}'\right]' + \ell'(\ell'+1) \int_{-1}^{+1} (\quad) = 0$$

$$\left[\ell(\ell+1) - \ell'(\ell'+1)\right] \int_{-1}^{+1} P_\ell P_{\ell'}\, dx = 0$$ Diklik bağıntısı

$\ell \neq \ell' \Rightarrow \int P_\ell P_{\ell'}\, dx = 0$

$\ell = \ell' \Rightarrow \int P_\ell^2\, dx \neq 0$

$$N_\ell \equiv \int_{-1}^{+1} \left[P_\ell(x)\right]^2 dx$$

$$= \frac{1}{2^{2\ell} (\ell!)^2} \int_{-1}^{+1} \underbrace{\frac{d^\ell}{dx^\ell} (x^2-1)^\ell \frac{d^\ell}{dx^\ell} (x^2-1)^\ell dx}_{\ell \text{ kez parçalı integrasyon}}$$

Legendre pol. için diklik bağıntısı

$$\int_{-1}^{+1} P_\ell(x) P_{\ell'}(x)\, dx = \frac{2}{2\ell+1} \delta_{\ell\ell'}$$

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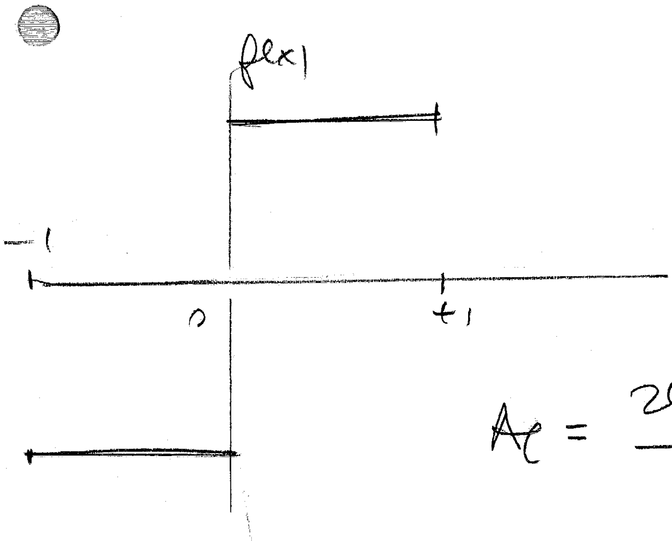
$[-1, +1]$ arasında LP'ı ortonormal fonk.'lar.

$$(-1 \leq x \leq +1)$$

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x) \qquad \int_{-1}^{+1} f(x) P_{l'}(x)\,dx = \sum_{l=0} \int_{-1}^{+1} A_l P_l P_{l'}\,dx$$

$$A_l = \frac{2l+1}{2} \int_{-1}^{+1} f(x) P_l(x)\,dx$$



$P_l(-x) = (-1)^l P_l(x)$ 'dir.

olsun.

$$A_l = \frac{2l+1}{2} \int_{-1}^{+1} f(x) P_l(x)\,dx$$

$$= \frac{2l+1}{2} \left\{ \int_{-1}^{0} (-1) P_l(x)\,dx + \int_{0}^{+1} (+1) P_l(x)\,dx \right\}$$

$$= \frac{2l+1}{2} \left\{ -\int_{0}^{1} P_l(-x)\,dx + \int_{0}^{1} P_l(x)\,dx \right\}$$

$$= \frac{2l+1}{2} \left[1 - (-1)^l \right] \int_{0}^{1} P_l(x)\,dx$$

$f \quad P_l$

tek) (çift) = tek Sim. aralıkta "0"

tek tek = çift $\neq$ "0"

$$= \begin{cases} 0 & l \text{ çift} \\ (2l+1) \int_0^1 dx\, P_l(x)\,dx & l \text{ tek.} \end{cases}$$

$$A_\ell = \frac{(2\ell+1)}{2^\ell \ell!}\int_0^1 dx \frac{d^\ell}{dx^\ell}(x^2-1)^\ell \qquad \ell \text{ tek.}$$

$$= \frac{2\ell+1}{2^\ell \ell!}\int_0^1 dx \frac{d}{dx}\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell$$

$$= \frac{2\ell+1}{2^\ell \ell!}\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell\Big|_0^1 = \left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}}\frac{(2\ell+1)(\ell-2)!!}{2\left[(\ell+1)/2\right]!}$$

$(\ell-2)!! = 1.3.5\ldots(\ell-2)$ eğer $\ell-2$ tek ise

$= 1.3.5\ldots(\ell-2-1)$ " $\ell-2$ çift ise

burada gerekli değil.

$$f(x) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(x)$$

$$= \frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \frac{11}{16}P_5(x) + \ldots$$

Rodriguez formülünden yararlanarak şunları yazabiliriz.

$$P'_{\ell+1} - P'_{\ell-1} - (2\ell+1)P_\ell = 0$$

$$(\ell+1)P_{\ell+1} - (2\ell+1)xP_\ell + \ell P_{\ell-1} = 0$$

$$P'_{\ell+1} - xP'_\ell - (\ell+1)P_\ell = 0 \; ; \; (x^2-1)P'_\ell - \ell x P_\ell + \ell P_{\ell-1} = 0$$

$$\int_{-1}^{+1} x\, P_\ell(x)\, P_{\ell'}(x)\, dx = \begin{cases} \dfrac{2(\ell+1)}{(2\ell+1)(2\ell+2)} & \ell' = \ell+1 \\[2ex] \dfrac{2\ell}{(2\ell-1)(2\ell+1)} & \ell' = \ell-1 \end{cases}$$

$$\int_{-1}^{+1} x^2\, P_\ell(x)\, P_{\ell'}(x)\, dx = \begin{cases} \dfrac{2(\ell+1)(\ell+2)}{(2\ell+1)(2\ell+3)(2\ell+5)} & \ell' = \ell+2 \\[2ex] \dfrac{2(\ell^2+2\ell-1)}{(2\ell-1)(2\ell+1)(2\ell+3)} & \ell' = \ell \end{cases}$$

3.3. Eksensel simetrili sınır-değer problemleri

aslında değişmez

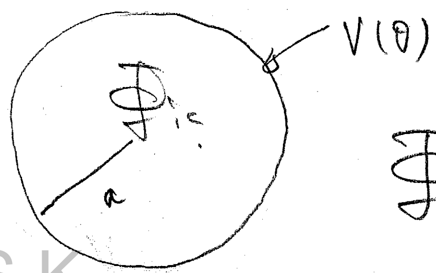
[z-de döndüğüm zaman pot. sabit, dönmelerden bağımsız, φ'ye bağımlılık yok.]

$\Phi \to \varphi$'den bağımsız $\Rightarrow m=0$

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + B_\ell r^{-(\ell+1)} \right] P_\ell(\cos\theta)$$

Orijin kapsanıyor ise, örn küre, içinde pot. i bulmak istiyorum, orijinde yük yok $\Rightarrow$ pot. burada sonlu olmalı $B_\ell = 0$

$r=0$



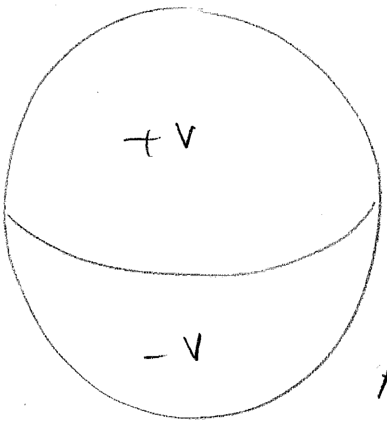
$$\Phi_{iç}(a,\theta) = V(\theta) = \sum_{\ell=0} A_\ell a^\ell P_\ell(\cos\theta)$$

$$A_\ell = \frac{2\ell+1}{2a^\ell} \int_{0\,(+1)}^{\pi\,(-1)} V(\theta)\, P_\ell(\cos\theta) \sin\theta \, d\theta$$

$\sin\theta\, d\theta = d(-\cos\theta)$

$\cos\theta = x.$

2.7. dahi



$$V(\theta) = \begin{cases} +V & 0 \le \theta \le \pi/2 \\ -V & \pi/2 \le \theta \le \pi \end{cases}$$

$$A_\ell = \frac{2\ell+1}{2a^\ell} \left[\int_0^{\pi/2} V\, P_\ell(\cos\theta) \sin\theta\, d\theta + \int_{\pi/2}^{\pi} (-V)\, P_\ell(\cos\theta) \sin\theta\, d\theta \right]$$

$$= \frac{2\ell+1}{\cancel{2}a^\ell} \cancel{2} V \int_0^{\pi/2} P_\ell(\cos\theta) \sin\theta\, d\theta = \frac{2\ell+1}{a^\ell} V \int_0^1 P_\ell(x)\, dx$$

$$\Phi(r,\theta) = V\left[\left(\frac{3}{2}\right)\left(\frac{r}{a}\right) P_1 - \left(\frac{7}{8}\right)\left(\frac{r}{a}\right)^3 P_3 + \dots \right]$$

$$\Phi(z=r) = V\left[1 - \frac{r^2 - a^2}{r\sqrt{r^2+a^2}} \right]$$

ekseni üzerinde bulmuştuk a^2/r^2'nin kuvvetlerine açalım.

$$\Phi(r,\theta) = \frac{V}{\sqrt{\pi}} \sum_{j=1}^{\infty} (-1)^{j-1} \frac{(2j-1/2)\Gamma(j-1/2)}{j!} \left(\frac{a}{r}\right)^{2j}$$

tek değerlerinin ($l = 2j-1$) ile karştığı görülür.

$$= \frac{V}{\sqrt{\pi}} \sum_{j=1}^{\infty} (-1)^{j-1} \frac{(2j-1/2)\Gamma(j-1/2)}{j!} \left(\frac{a}{r}\right)^{2j} P_{2j-1}(\cos\theta)$$

2.27 ve 3.36. ile aynı.

Dış bölgede çözüm istenir ise, $r \to \infty$ da $\Phi \to$ sonlu olmalı (0)

$$\Phi(r\to\infty) \to r^{-(l+1)} \to 0 \quad \text{fakat } r^{l} \to \infty \Rightarrow A_l = 0$$

$$\Phi_{Dış}(r,\theta) = \sum_{l} B_l \frac{1}{r^{l+1}} P_l(\cos\theta)$$

$$V(a,\theta) = \sum B_l \frac{1}{a^{l+1}} P_l(\cos\theta)$$

$$B_l = \frac{(2l+1)\, a^{l+1}}{2} \int_{-1}^{+1} V(\theta,\phi)\, P_l \, d(\cos\theta)$$

r'yi z üzerine gelip, burada pot.'i değerlendirelim!

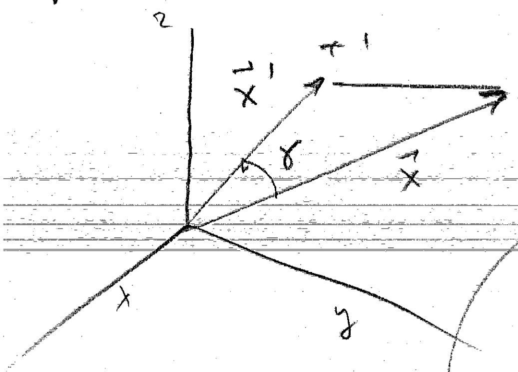
$\theta = 0$

$$\Phi(r=z) = \sum \left(A_\ell z^\ell + B_\ell z^{-(\ell+1)}\right)$$

daha sonra z yerine r alıp r ye genişletebiliriz.

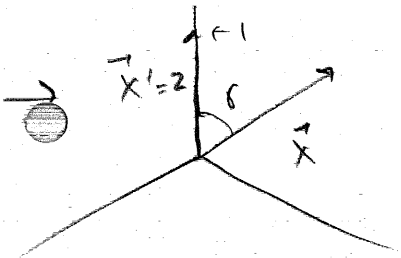
r^ℓ leri B_ℓ ile $r^{-\ell}$ 'leri $P_{\ell-1}$ ile çarparım.

[bu yöntemi daha çok z'ye göre sim. yük dağılımlarında kullanacağız.]



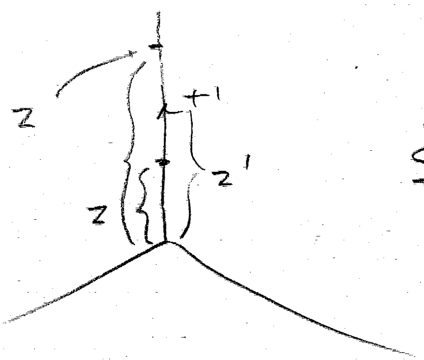
$\frac{1}{|\vec{x}-\vec{x}'|}$ 'yi LP cinsinden yazalım.

koord. dön. ü ile $\vec{x}'$ 'yü z ile çakıştırabiliriz.



$$= \sum \left(A_\ell r^\ell + B r^{-(\ell+1)}\right) P_\ell(\cos\gamma)$$

$\vec{x}'$ 'i buradaki z ekseninde düşünürsem



$$\Phi(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} \rightarrow \Phi(z) = \frac{1}{|z-z'|}$$

$z < z'$, $z > z'$ olabilir.

$$z < z' \Rightarrow \frac{1}{|z-z'|} = \frac{1}{z'}\frac{1}{\left|1-\frac{z}{z'}\right|} = \frac{1}{z'}\left(1-\frac{z}{z'}\right)^{-1}$$ Taylor serisine açalım.

$$= \frac{1}{z'}\left(1+\frac{z}{z'}+\frac{(-1)(\cancel{-2})}{\cancel{2!}}\left(\frac{z}{z'}\right)^2+\frac{(-1)(-2)(\cancel{-3})}{\cancel{3!}}\left(-\frac{z}{z'}\right)^3+\dots\right)$$

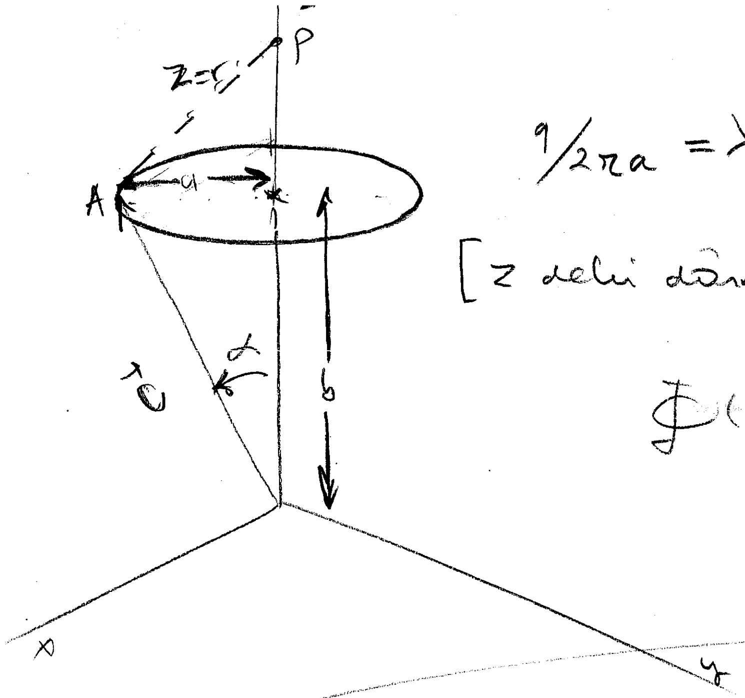
$$= \frac{1}{z'}\sum_{\ell=0}^{\infty}\left(\frac{z}{z'}\right)^{\ell} \qquad z<z' \qquad z=r_< \quad z'=r_>$$

$$= \frac{1}{z}\sum_{\ell=0}^{\infty}\left(\frac{z'}{z}\right)^{\ell} \qquad z>z' \qquad z'=r_< \quad z=r_>$$

$$\frac{1}{|z-z'|}=\sum_{\ell=0}^{\infty}\frac{r_<^{\ell}}{r_>^{\ell+1}} \qquad \text{ozaman,}$$

$$\frac{1}{|\vec{x}-\vec{x}'|}=\sum_{\ell=0}^{\infty}\frac{r_<^{\ell}}{r_>^{\ell+1}}P_\ell(\cos\gamma)$$

$$=\begin{cases} r>r' & \dfrac{1}{|\vec{x}-\vec{x}'|}=\displaystyle\sum\frac{r'^{\ell}}{r^{\ell+1}}P_\ell(\cos\gamma) \\[2em] r<r' & =\displaystyle\sum\frac{r^{\ell}}{r'^{\ell+1}}P_\ell(\cos\gamma)\end{cases}$$



$q/2\pi a = \lambda$

[z dahi dönmelere göre sim. simetri gözel değildir.]

$$\Phi(z) = \frac{q}{|AP|} = \frac{q}{|\vec{z} - \vec{c}|}$$

$$= q \sum_{l=0}^{\infty} \frac{c^l}{z^{l+1}} P_l(\cos\alpha) \qquad c < z$$

$$= q \sum_{l=0}^{\infty} \frac{z^l}{c^{l+1}} P_l(\cos\alpha) \qquad c > z$$

P noktasını dışarı çıkarırsam,

$$\Phi(r,\theta) = q \begin{cases} \sum_{l=0}^{\infty} \left(c^l / r^{l+1} \right) P_l(\cos\alpha) P_l(\cos\theta) & c < r \\ \sum_{l=0}^{\infty} \left(r^l / c^{l+1} \right) P_l(\cos\alpha) P_l(\cos\theta) & c > r \end{cases}$$

İçi dolu olsa idi; $dq = 2\pi c \sin\alpha$

$$c = b / \cos\alpha$$

$$\Phi(z) = \int_0^{\alpha_0} \frac{2\pi c \sin\alpha}{|\vec{z} - \vec{c}|} d\tau$$

$$\left(z^2+c^2-2cz\cos\alpha\right)^{1/2}=\left(z^2+\frac{b^2}{\cos^2\alpha}-2zb\right)^{1/2}$$

3.5.– Bağlı Legendre fonk. ları ve $Y_{\ell m}(\theta,\varphi)$ küresel harmonikleri

- Eksensel simetrili problemlerden ziyade, genel pot. problemi φ 'değişimlerini de içereceğinden $m \neq 0$ 'dır.
- Legendre fonk. larının sonlu olabilmesi için ℓ pozitif tamsayı olmalı m de buna bağlı olarak

$$-\ell,\ -(\ell-1),\ \dots\ 0,\ \dots,\ (\ell-1),\ \ell$$

değerlerini alabilir.

$$P_\ell^m(\underbrace{\cos\theta}_{x})=(-1)^m\,(1-x^2)^{m/2}\,\frac{d^m}{dx^m}P_\ell(x)$$

Adi Legendre dif. denk. i

$$(1-x^2)P_\ell''-2xP_\ell'+\ell(\ell+1)P_\ell=0$$

m kere x 'e göre türetelim. $u\equiv\dfrac{d^m P}{dx^m}$ diyelim

$$\frac{d^m}{dx^m}[AB]=\sum_{s=0}^{m}\underbrace{\frac{m!}{(m-s)!\,s!}}_{\binom{m}{s}}\frac{d^{m-s}}{dx^{m-s}}A\,\frac{d^s}{dx^s}B \quad \text{Leibniz formülü}$$

$$-2xP_\ell'' + (1-x^2)P_\ell''^{+1} - 2P_\ell' - 2xP_\ell'^{+1} + \ell(\ell+1)P_\ell^{+1} = 0$$

$$-2P_\ell'' - 2xP_\ell''^{+1} - 2xP_\ell''^{+1} + (1-x^2)P_\ell''^{+2} - 2P_\ell'^{+1} - 2P_\ell'^{+1} - 2xP_\ell'^{+2}$$
$$+\ell(\ell+1)P_\ell^{+2} = 0$$

$\vdots$

$$(1-x^2)u'' - 2x(m+1)u' + (\ell-m)(\ell+m+1)u = 0$$

$v = (1-x^2)^{m/2}\, u$ tanımlarsak,

$u = (1-x^2)^{-m/2}\, v$, u', u'' yerine koy!

$$(1-x^2)v'' - 2xv' + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]v = 0$$

$$P_\ell^m(\cos\theta) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}\frac{d^{m+\ell}}{dx^{m+\ell}}(x^2-1)^\ell$$

$$P_\ell^{-m}(\cos\theta) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\cos\theta)$$

$$\int_{-1}^{+1} P_\ell^m(x) P_{\ell'}^m(x)\,dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell\ell'}$$

BS K

$$\Phi(r,\theta,\varphi) = \sum_{l}\sum_{m} A_{lm} R_l(r) \underbrace{P_l^m(x)\, Q_m(\varphi)}$$

$$Y_l^m(\theta,\varphi)$$

birim küre yüzeyi üzerinde tanımlanmış bütün boyutlu küresel harmonikler.

$$Y_l^m(\theta,\varphi) = \left(\frac{2l+1}{4\pi}\right)^{1/2} \left(\frac{(l-m)!}{(l+m)!}\right)^{1/2} P_l^m(\cos\theta)\, e^{im\varphi}$$

$$\int_{-1}^{+1} d(\cos\theta) \int_0^{2\pi} d\varphi\; Y_l^m(\theta,\varphi)\, Y_{l'}^{m'^*}(\theta,\varphi) = \delta_{mm'}\delta_{ll'}$$

$$\iint Y_l^m(\theta,\varphi)\, Y_{l'}^{m'^*}(\theta,\varphi)\, d\Omega = \delta_{mm'}\delta_{ll'}$$

küre yüzeyi üzerinde herhangi bir $g(\theta,\varphi)$ fonk.'u bunlar cinsinden

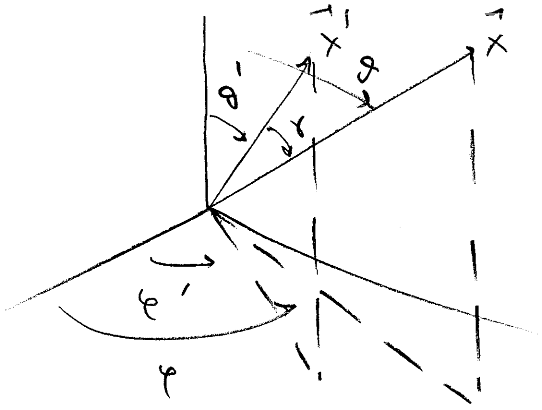
$$g(\theta,\varphi) = \sum_{m=-l}^{+l} \sum_{l=0}^{\infty} A_{lm} Y_l^m(\theta,\varphi)$$

$$\iint Y_{l'}^{m'^*}\, g(\theta,\varphi)\, d\Omega = \sum\sum A_{lm} \iint \underbrace{Y_{lm} Y_{l'm'}^*}\, d\Omega$$

$$A_{lm} = \int_{-1}^{+1}\int_0^{2\pi} d\Omega\; Y_l^{m^*}\, g(\theta,\varphi)$$

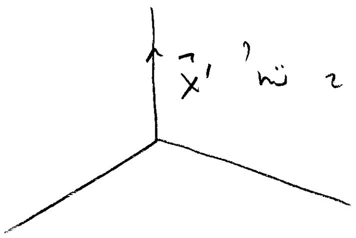
$$\Phi(r,\theta,\varphi) = \sum_{l}\sum_{m} \left[A_{lm} r^l + B_{lm} r^{-(l+1)}\right] Y_{lm}(\theta,\varphi)$$

3.6. Küresel Harmonikler için toplama teoremi



$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$$

$$P_l(\cos\gamma) = \sum_{l'}^{\infty}\sum_{m=-l'}^{+l'} A_{l'm}(\theta',\varphi')\,Y_{l'm}(\theta,\varphi)$$



$\vec{x}'$ 'nü z-ekseni ile çakıştırırsak $\gamma \to \theta$ olur. Bu durumda

$$\nabla'^2 P_l(\cos\gamma) + \frac{l(l+1)}{r^2}P_l(\cos\gamma) = 0$$

denklemini sağlar.

$\nabla'^2 = \nabla^2$ değişmezliği var. ($\vec{\nabla}\cdot\nabla\psi$ op.'ü yani skaler çarpım dönmeler altında korunurlar)

$\Downarrow$

$$\nabla^2 P_l(\cos\gamma) + \frac{l(l+1)}{r^2}P_l(\cos\gamma) = 0$$ yani P_l sadece l'nin fonk.u.

$$\Rightarrow P_l(\cos\gamma) = \sum_{m=-l}^{+l} A_{lm}(\theta',\varphi')\,Y_{lm}(\theta,\varphi)$$

$$A_{lm}(\theta',\varphi') = \int d\Omega\, Y^*_{lm}\, P_l(\cos\gamma)$$

$$= \frac{4\pi}{2l+1}\left\{Y^*_{lm}\left[\theta(\gamma,\beta),\varphi(\gamma,\beta)\right]\right\}_{\gamma=0}$$

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{+\ell} Y_{\ell m}^*(\vartheta', \varphi') Y_{\ell m}(\vartheta, \varphi)$$

$\gamma = 0 \Rightarrow \cos\gamma = 1 \quad \vartheta' = \vartheta, \ \varphi' = \varphi$ damit

$$1 = \frac{4\pi}{2\ell+1} \sum_m |Y_{\ell m}(\vartheta, \varphi)|^2$$

$$\Rightarrow \sum_m |Y_{\ell m}(\vartheta, \varphi)|^2 = \frac{2\ell+1}{4\pi}$$

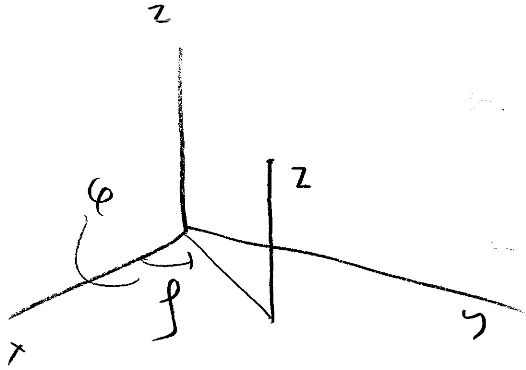
$\ell = 2 \Rightarrow m = -2, -1, 0, +1, +2$

$$|Y_2^{-2}|^2 + |Y_2^{-1}|^2 + |Y_2^0|^2 + \dots = \frac{5}{4\pi}$$

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma)$$

$$= \sum_\ell \sum_{m=-\ell}^{+\ell} \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_\ell^{m*}(\vartheta', \varphi') Y_\ell^m(\vartheta, \varphi)$$

3.7. Silindirik koordinatlarda Laplace denk.'i, Bessel Fonksiyonları:



$$\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Phi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Phi(\rho, \varphi, z) = R(\rho)\, Q(\varphi)\, Z(z)$$

$$\underbrace{\frac{1}{R}\frac{d^2R}{d\rho^2} + \frac{1}{R\rho}\frac{dR}{d\rho} + \frac{1}{\rho^2 Q}\frac{d^2Q}{d\varphi^2}}_{-k^2} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{+k^2} = 0$$

$$Z(z) = Ae^{kz} + Be^{-kz}$$

$$\underbrace{k^2\rho^2 + \frac{\rho^2}{R}R'' + \frac{\rho}{R}\rho'}_{+\nu^2} + \underbrace{\frac{1}{Q}\frac{d^2Q}{d\varphi^2}}_{-\nu^2} = 0$$

$$Q(\varphi) = Ce^{i\nu\varphi} + De^{-i\nu\varphi}$$

$$R'' + \frac{1}{\rho}R' + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R = 0$$

$$k\rho = x \Rightarrow \frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0$$ Bessel Dif. Denk.'i

BS K

$$R(x) = \sum_{j=0}^{\omega} a_j \, x^{j+\alpha}$$

$$\sum_{j=0} a_j \,(\alpha+j)(\alpha+j-1)\, x^{\alpha+j-2} + \sum_{j=0} a_j (\alpha+j)\, x^{\alpha+j-2}$$

$$+ \sum_{j=0} a_j \, x^{\alpha+j} - \nu^2 \sum_{j=0} a_j \, x^{\alpha+j-2} = 0$$

$$\sum_{j=0}^{\omega} a_j \left\{ (\alpha+j)(\alpha+j-1) + (\alpha+j) - \nu^2 \right\} x^{\alpha+j-2} + \sum_{j=0}^{\omega} a_j x^{\alpha+j} = 0$$

$$\underbrace{a_0 \Big[\overbrace{\alpha(\alpha-1)+\alpha-\nu^2}^{\alpha^2-\nu^2} \Big] x^{\alpha-2} + a_1 \Big[\overbrace{(\alpha+1)\alpha + (\alpha+1) - \nu^2}^{(\alpha+1)^2-\nu^2} \Big] x^{\alpha-1}}_{\#}$$

$$+ \sum_{j=2} \{ \qquad \} \, x^{\alpha+j-2} + \sum_{j=0} a_j \, x^{\alpha+j} = 0$$

$$\nearrow j \to j+2$$

$$\# + \sum_{j=0}^{\omega} \left\{ a_{j+2} \left[(\alpha+j+2)(\alpha+j+1) + (\alpha+j+2) - \nu^2 \right] + a_j \right\} x^{\alpha+j} = 0$$

$$a_0 \left(\alpha^2 - \nu^2 \right) = 0$$

$$a_1 \left[(\alpha+1)^2 - \nu^2 \right] = 0$$

$$a_{j+2} = a_j \, \frac{1}{\nu^2 - (\alpha+j+2)^2}$$

$a_0 \neq 0 \qquad \alpha^2 = \nu^2 \qquad \alpha = \pm \nu$

$a_1 = 0$

$$a_{j+2} = a_j \frac{1}{\alpha^2 - (\alpha+j+2)^2} = \frac{-1}{(j+2)(j+2+2\alpha)} a_j$$

$$\Rightarrow a_{2j} = \frac{-1}{2j(2j+2\alpha)} a_{2j-2} = \frac{-1}{4j(\alpha+j)} a_{2j-2} = \frac{-1}{4j(\alpha+j)} a_{2(j-1)}$$

$$a_{2(j-1)} = \frac{-1}{4(j-1)(\alpha+j-1)} a_{2(j-2)}$$

$$a_{2(j-2)} = \frac{-1}{4(j-2)(\alpha+j-2)} a_{2(j-3)} \quad \cdots$$

$$a_{2j} = \frac{-1}{4j(\alpha+j)} \cdot \frac{-1}{4(j-1)(\alpha+j-1)} \cdots \frac{-1}{4\cdot 1(\alpha+1)} \frac{\alpha!}{\alpha!} a_0$$

$$= \frac{(-1)^j}{4^j j! (\alpha+j)!} a_0 = \frac{(-1)^j \Gamma(\alpha+1)}{4^j j! \Gamma(\alpha+j+1)} a_0$$

$$a_0 = \frac{1}{2^\alpha \Gamma(\alpha+1)}$$

$\alpha = +\nu \qquad J_\nu(x) = ?$

$$R = \sum_{j=0}^{\infty} a_j x^{\alpha+j}$$ tek j'ler sıfır.

$$= \sum_{j=0} a_{2j} x^{\nu+2j} = \sum_j \frac{(-1)^j x^{\nu+2j}}{2^{\nu+2j} j! \Gamma(\nu+j+1)}$$

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j}$$ I. tür Bessel fonk.

$$\alpha=-\nu \Rightarrow J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j-\nu+1)} \left(\frac{x}{2}\right)^{2j}$$

bunlar birbirinden bağımsız ν tamsayı $\Rightarrow$ $(\nu=m)$ birbirine bağlıdırlar.

$$J_{-m}(x) = (-1)^m J_m(x)$$

$$J_{-m}(x) = \left(\frac{x}{2}\right)^{-m} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j-m+1)} \left(\frac{x}{2}\right)^{2j}$$

$j=m$'ye kadar olan değerlerde $\Gamma \to \infty$'dur. $j=m$ de $(\Gamma(1))$ olur ve pozitif değerler almaya başlar

$j=m$'den

$j=k+m$ ya da $j \to j+m$

$$J_{-m}(x) = \left(\frac{x}{2}\right)^{-m} \sum_{j=0}^{\infty} \frac{(-1)^{j+m}}{(j+m)!\,\Gamma(j+1)} \left(\frac{x}{2}\right)^{2(j+m)}$$

$$= \left(\frac{x}{2}\right)^{m} \sum_{j=0} \frac{(-1)^{j+m}}{\underbrace{(j+m)!}_{\Gamma(j+m+1)}\,\underbrace{\Gamma(j+1)}_{j!}} \left(\frac{x}{2}\right)^{2j}$$

$$=(-1)^m J_m(x)$$

$$N_\nu = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin\nu\pi} \qquad \text{II. tür BF}$$

$$\left.\begin{aligned} H_\nu^{(1)} &= J_\nu + iN_\nu \\ H_\nu^{(2)} &= J_\nu - iN_\nu \end{aligned}\right\} \qquad \begin{aligned} \Omega_{\nu-1} + \Omega_{\nu+1} &= \frac{2\nu}{x}\Omega_\nu \\ \Omega_{\nu-1} - \Omega_{\nu+1} &= 2\Omega_\nu'(x) \end{aligned}$$

Ω : J_ν, N_ν, H den herhangi biri.

$$x \ll 1 \Rightarrow \begin{cases} J_\nu(x) \to \dfrac{1}{\Gamma(\nu+1)}\left(\dfrac{x}{2}\right)^\nu \\ N_\nu(x) \to \begin{cases} \dfrac{2}{\pi}\left[\ln\left(\dfrac{x}{2}\right) + 0.5772 + \ldots\right] & \nu = 0 \\ -\dfrac{\Gamma(\nu)}{\pi}\left(\dfrac{2}{x}\right)^\nu & \nu \neq 0 \end{cases} \end{cases}$$

$$x \gg 1 \Rightarrow \quad J_\nu(x) \to \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$$N_\nu(x) \to \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$X_{\nu n}$: $J_\nu(x)$'in n. ci kökü olsun $J_\nu(X_{\nu n}) = 0$

ilk üç kök,

		n=1	2	3
$\nu=0$	X_{0n}	2.405	5.520	8.654
$\nu=1$	X_{1n}	3.832	7.016	10.175

$\leftarrow \sqrt{\rho}\, J_\nu(X_{\nu n}\rho/a)$

$0 \leq \rho \leq a$ aralığında bu fonk'lar tam ve dik bir fonk. kümesi oluştururlar.

Final.

$x = k\rho$

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)\right]+\left(\frac{x_{\nu n}^2}{a^2}-\frac{\nu^2}{a^2}\right)J_\nu\left(\frac{x_{\nu n}\rho}{a}\right)=0$$

$\rho J_\nu(x_{\nu n'}\rho/a)$ ile çarpıp

$$\int_0^a J_\nu\left(\frac{x_{\nu n'}\rho}{a}\right)\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J(x_{\nu n}\rho/a)\right]d\rho$$

$$+\int_0^a J_\nu(x_{\nu n}\rho/a)J_\nu(x_{\nu n'}\rho/a)\left(\frac{x_{\nu n}^2}{a^2}-\frac{\nu^2}{\rho^2}\right)d\rho=0$$

$$\rho J_\nu(x_{\nu n'}\rho/a)\frac{dJ_\nu}{d\rho}\Big|_0^a-\int\rho\frac{d}{d\rho}J_\nu(\;)\frac{d}{d\rho}J_\nu(\;)'d\rho$$

$$-\int\rho\frac{d}{d\rho}J_\nu(x_{\nu n'}\rho/a)\frac{d}{d\rho}J_\nu(x_{\nu n}\rho/a)\,d\rho+(\;)=0$$

$n' \to n$ yapıp taraf tarafa çıkartırsak,

$$\frac{[x_{\nu n}^2-x_{\nu n'}^2]}{a^2}\int_0^a\rho J_\nu(x_{\nu n}\rho/a)J_\nu(x_{\nu n'}\rho/a)\,d\rho=0$$

$n\neq n' \Rightarrow x_{\nu n}^2-x_{\nu n'}^2\neq 0$

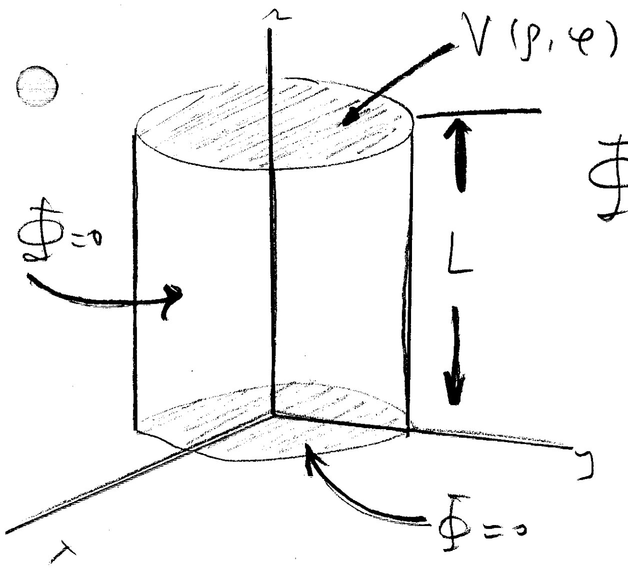
$$\int_0^a\rho J_\nu(x_{\nu n}\rho/a)J_\nu(x_{\nu n'}\rho/a)\,d\rho=\frac{a^2}{2}\left[J_{\nu+1}(x_{\nu n})\right]^2\delta_{nn'}$$

BS K

$$f(\rho) = \sum_{n=1}^{\infty} A_{\nu n} J_\nu\left(\frac{x_{\nu n}\rho}{a}\right)$$

$$A_{\nu n} = \frac{2}{a^2\left[J_{\nu+1}(x_{\nu n})\right]^2} \int_0^a \rho\, f(\rho)\, J_\nu\left(x_{\nu n}\frac{\rho}{a}\right) d\rho$$

3.8. Silindirik koordinatlarda sınır-değer problemleri



$$\Phi = \sum_{mn} \left(A_{mn} J_m(k_{mn}\rho) + B_{mn} N_m(k_{mn}\rho)\right)$$
$$\times \left(C_{mn} \sin m\varphi + D_{mn} \cos m\varphi\right)$$
$$\times \left(E_{mn} e^{k_{mn} z} + F_{mn} e^{-k_{mn} z}\right)$$

$z = 0$ da $\Phi = 0$ $\quad E + F = 0 \quad F = -E \rightarrow \operatorname{sh} k_{mn} z$

$\rho = 0$ da çözüm sonlu olmalı $\quad B = 0 \quad x \ll 1 \Rightarrow J_\nu(x) \rightarrow \dfrac{1}{\Gamma(\nu+1)}\left(\dfrac{x}{2}\right)^\nu \rightarrow 0$

$N_\nu \rightarrow \infty$

$$\Phi(\rho, \varphi, z) = \sum_{m,n} \left(A_{mn} \sin m\varphi + B_{mn} \cos m\varphi\right) J_m(k_{mn}\rho) \operatorname{sh} k_{mn} z$$

$\rho = a$ da $\Phi = 0 \Rightarrow J_m(k_{mn} a) = 0 \qquad k_{mn} a = X_{mn}$

$$k_{mn} = \frac{X_{mn}}{a}$$

$$\Phi(\rho,\varphi,z) = \sum_{m,n} \left(A_{mn}\sin m\varphi + B_{mn}\cos m\varphi\right) J_m\left(\frac{X_{mn}}{a}\rho\right)$$
$$\times\, Sh\left(\frac{X_{mn}}{a} z\right)$$

$z=L$ de $\Phi = V(\rho,\varphi)$

$$\sum_{mn} \underbrace{\left(A_{mn}\sin m\varphi + B_{mn}\cos m\varphi\right)}_{\#} J_m\left(\frac{X_{mn}\rho}{a}\right) Sh\left(\frac{X_{mn}}{a}L\right) = V(\rho,\varphi)$$

$$\int_0^a \int_0^{2\pi} \sin q\varphi \, J_m\left(\frac{X_{mn'}}{a}\rho\right) V(\rho,\varphi)\,\rho\, d\rho\, d\varphi$$

$$= \int_0^a \int_0^{2\pi} \sum_{mn} (\#)\, J_m\left(\frac{X_{mn}\rho}{a}\right) Sh\left(\frac{X_{mn} z}{a}\right) \rho\, d\rho\, d\varphi$$

$$= \pi A_{qn'} \frac{a^2}{2} J_{m+1}^2 \left(X_{mn'}\right)$$

$$A_{mn} = \frac{2}{J_{m+1}^2 \pi a^2 \, Sh\left(\frac{X_{mn}L}{a}\right)} \int_0^a \rho\, d\rho \int_0^{2\pi} d\varphi\, V(\rho,\varphi) \sin m\varphi\, J_m\left(\frac{X_{mn}\rho}{a}\right)$$

B_{mn} için her iki tarafı $\cos q\varphi$ ile çarpar.

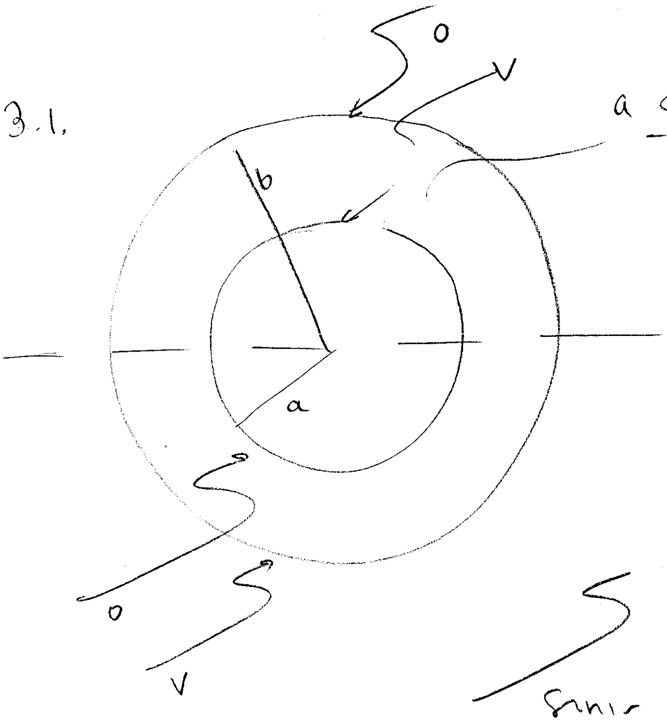
Not $z=L$ de $\Phi = 0$ olsaydı $\underbrace{E e^{kL} + F e^{-kL}}_{} = 0$

$$F = -e^{2kL} E$$

$$= E e^{kL}\left(e^{k(z-L)} - e^{-k(z-L)}\right) = E e^{kL}\, Sh\, k(z-L) \quad z=L \text{ de sıfır.}$$

$$= A\, Sh\, k(z-L) \text{ olurdu.}$$

3.1.



$a \le r \le b$ bölgesindeki pot.'i Legendre çok terimlerinin bir serisi olarak bul.

$$\Phi(a,\theta)=\begin{cases} V & 1 \ge \cos\theta \ge 0 \\ 0 & 0 \ge \cos\theta \ge -1 \end{cases}$$

$$\Phi(b,\theta)=\begin{cases} 0 & 1 \ge \cos\theta \ge 0 \\ V & 0 \ge \cos\theta \ge -1 \end{cases}$$

Sınır koşulları

verilen çözüm φ'den bağımsız, eksensel simetriye sahip

$$\Phi(r,\theta)=\sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta)$$

Sınır koşullarından

$$A_l a^l + B_l \frac{1}{a^{l+1}} = \frac{2l+1}{2}\int_{-1}^{+1} \Phi(a,\theta)\, P_l(x)\,dx$$

$$= \frac{2l+1}{2} V \int_{0}^{1} P_l(x)\,dx$$

$$A_l b^l + B_l \frac{1}{b^{l+1}} = \frac{2l+1}{2}\int_{-1}^{+1} \Phi(b,\theta)\, P_l(x)\,dx$$

$$= \frac{2l+1}{2} V \int_{-1}^{0} P_l(x)\,dx$$

$$(2l+1)P_l(x) = P'_{l+1}(x) - P'_{l-1}(x)$$

$$\int_0^1 (2l+1)P_l(x)dx = \int_0^1 \left[P'_{l+1}(x) - P'_{l-1}(x) \right] dx$$

$$= P_{l+1}(1) - P_{l+1}(0) - P_{l-1}(1) + P_{l-1}(0)$$

$$= \begin{cases} 0 & l \text{ çift} \\ 1 & l = 0 \\ -P_{l+1}(0) + P_{l-1}(0) & l \text{ tek} \end{cases}$$

l tek $\Rightarrow l \to 2n+1$

$$(4n+3)\int_0^1 P_{2n+1}(x)dx = P_{2n}(0) - P_{2n+2}(0)$$

$$= \frac{(-1)^n (2n)!}{2^{2n}[(n!)!]^2} - \frac{(-1)^{n+1}(2n+2)!}{2^{2n+2}[(n+1)!]^2}$$

$$= \frac{(-1)^n (2n)!}{2^{2n}(n!)^2}\left[1 + \frac{2n+1}{2n+2}\right]$$

benzer şekilde

$$(4n+3)\int_{-1}^{0} P_{2n+1}(x)dx = \frac{(-1)^{n+1}(2n)!}{2^{2n}(n!)^2}\left[1 + \frac{2n+1}{2n+2}\right]$$

BS K

$$\left.\begin{aligned} A_0 + B_0 \frac{1}{a} &= \frac{V}{2} \\ A_0 + B_0 \frac{1}{b} &= \frac{V}{2} \end{aligned}\right\} \Rightarrow A_0 = \frac{V}{2} \qquad B_0 = 0$$

$$A_1 = -\frac{3V}{4} \frac{a^2+b^2}{b^3-a^3} \qquad B_1 = \frac{3V}{4} \frac{a^2 b^2 (a+b)}{b^3-a^3}$$

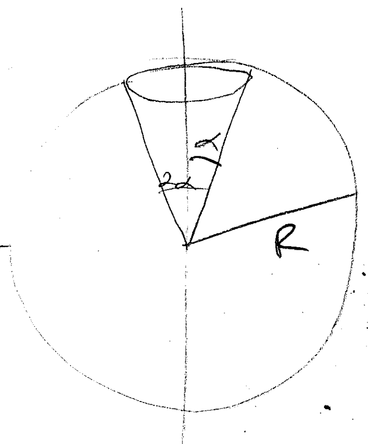
$$A_3 = \frac{7V}{16} \frac{a^4+b^4}{b^7-a^7} \qquad B_3 = -\frac{7V}{16} \frac{a^4 b^4 (a^3+b^3)}{b^7-a^7} \quad \dots$$

$$\Phi(r,\theta) = \frac{V}{2} - \frac{3V}{4} \frac{a^2+b^2}{b^3-a^3} r P_1(\cos\theta) + \frac{3V}{4} \frac{a^2 b^2 (a+b)}{b^3-a^3} \frac{P_1(\cos\theta)}{r^2}$$

$$+ \dots$$

$$\lim_{a \to 0}, \ \lim_{b \to \infty} \to \Phi \to ?$$

3.2.

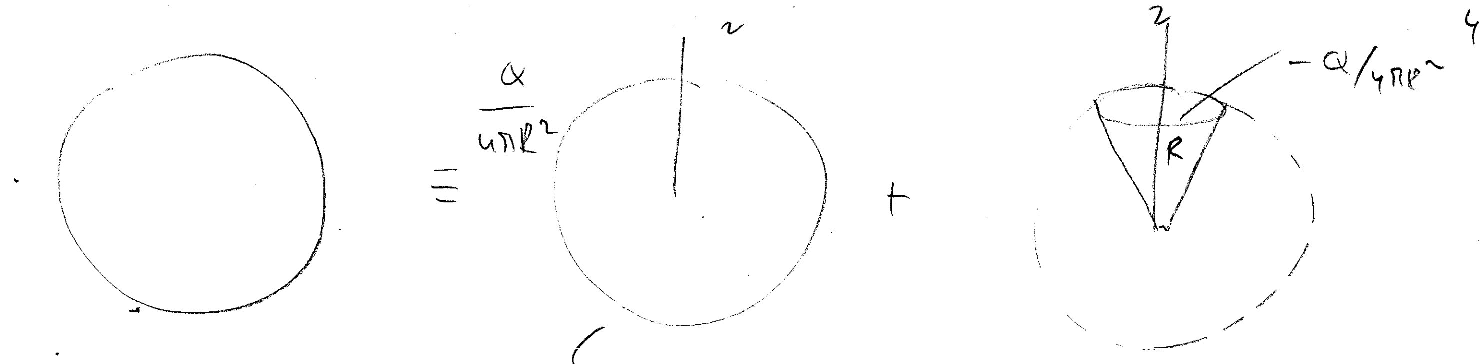


$$\frac{Q}{4\pi R^2} = \sigma$$

$$\Phi_{iç} = \sum_{l=0}^{\infty} A_l r^l P_l$$

$$\Phi_{dış} = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l$$

$r = R$ de $\Phi_{iç} = \Phi_{dış}$ $\qquad B_l = A_l R^{2l+1}$



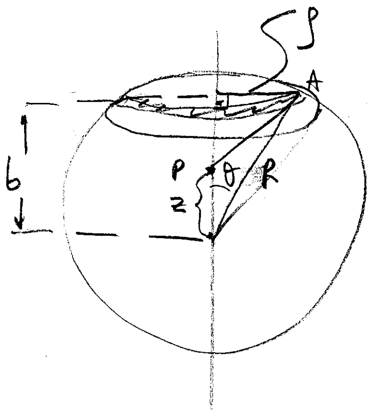
$\int$ sbt bir pot. verir $= \dfrac{Q}{R}$

z-eksenine göre sim. bir yük dağılımı. $z = R$ yazarım

$\theta = 0$ z-ekseni üzerinde çözüm verir.

$$\Phi(z) = \sum_{\ell} \left[A_\ell z^\ell + B_\ell z^{-(\ell+1)} \right]$$

$z = R$ koyar $[\;\;]\, P_\ell(\cos\theta)$ ile çarparım.



$$b = R\cos\theta \qquad AP = \sqrt{\rho^2 + (b - z)^2}$$

$$\rho = R\sin\theta \qquad\quad = \sqrt{R^2 + z^2 - 2Rz\cos\theta}$$

şeridin üzerindeki yük $= \left(-\dfrac{Q}{4\pi R^2}\right) \times 2\pi \underbrace{\rho}_{R\sin\theta} R\, d\theta = -\dfrac{Q}{{}_2\cancel{4\pi R^2}} \cancel{2\pi R^2} \sin\theta\, d\theta$

z deki pot. $\Phi(z) = \displaystyle\int_0^\alpha \frac{-\frac{Q}{2}\sin\theta\, d\theta}{\sqrt{R^2 + z^2 - 2Rz\cos\theta}}$

$$= \frac{Q}{2} \int_1^{\cos\alpha} \frac{d(\cos\theta)}{\sqrt{R^2 + z^2 - 2Rz\cos\theta}} = \frac{Q}{2} \int_1^{\cos\alpha} \frac{d(\cos\theta)}{|\vec{z} - \vec{R}|}$$

bunu seriye açalım.

$$\Phi(z) = \frac{Q}{2}\int_1^{\cos\alpha} d(\cos\theta) \sum_{l=0}^{\infty} \frac{z^l}{R^{l+1}} P_l(\cos\theta)$$

$$= \frac{Q}{2}\sum_l \frac{z^l}{R^{l+1}} \int_1^{\cos\alpha} dx \underbrace{P_l(x)}_{\frac{1}{2l+1}\left[P'_{l+1} - P'_{l-1}\right]}$$

$$= \frac{Q}{2}\sum_l \frac{1}{2l+1}\frac{z^l}{R^{l+1}}\Big[P_{l+1}(x) - P_{l-1}(x)\Big]_1^{\cos\alpha}$$

$$= \frac{Q}{2}\sum_l \frac{1}{2l+1}\frac{z^l}{R^{l+1}}\Big[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\Big]$$

z ekseni üzerindeki çözüm (P deki

P noktasının dışına çıkarsam $z=r$ koyar $P_l(\cos\theta)$ ile çarparım.

İç bölgede

$$\Phi_{iç}(r,\theta) = \frac{Q}{2}\sum \frac{1}{2l+1}\frac{r^l}{R^{l+1}}\Big[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\Big]P_l(\cos\theta)$$

dış bölgede $\dfrac{z^l}{R^{l+1}} \longrightarrow \dfrac{R^{l+1}}{z^l}$ yazarım.

$\alpha \to 0 \Rightarrow E = 0$

$P_2(\cos\alpha) = 1 \quad E_r = 0$

$E_\theta = 0$

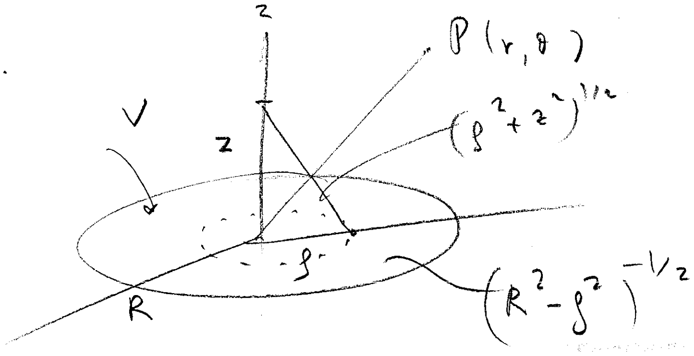
b) $\vec{E}_{iç} = -\vec{\nabla}\Phi_{iç}$ $\qquad E_r = -\dfrac{\partial \Phi_{iç}}{\partial r}$ $\qquad r=0$ da

$\vec{E}_{dış} = -\vec{\nabla}\Phi_{dış}$ $\qquad E_\theta = -\dfrac{1}{r}\dfrac{\partial \Phi_{iç}}{\partial \theta}$

$$E_r\Big|_{r=0} = -\frac{Q}{6}\Big[P_2(\cos\alpha) - 1\Big]\frac{\cos\theta}{R^2}$$

$$E_\theta\Big|_{r=0} = -Q/6R^2\Big[P_2(\cos\alpha) - 1\Big]$$

3.3.



$r > R$ için $\Phi(r,\theta,\varphi) = ?$

$$\sigma = A\,(R^2-\rho^2)^{-1/2}$$

z-ekseni üzerindeki pot. $\displaystyle \Phi(z) = \int_0^R \frac{A\, 2\pi\rho\, d\rho \,/\, (R^2-z^2)^{+1/2}}{(\rho^2+z^2)^{1/2}}$

$$\rho^2 = u \Rightarrow \Phi(z) = \int_0^{R^2} \frac{A\pi\, du}{\sqrt{-u^2+(R^2-z^2)u+z^2R^2}}$$

$$= -A\pi \sinh^{-1}\left(\frac{R^2-z^2-2u}{R^2+z^2}\right)\Bigg|_0^{R^2} = A\pi\left[\sinh^{-1}(1) + \sinh^{-1}\left(\frac{R^2/z^2-1}{R^2/z^2+1}\right)\right]$$

$$\Phi(z=0) = V = 2\pi A \sinh^{-1}(1)$$

$$A = V/2\pi \sinh^{-1} 1$$

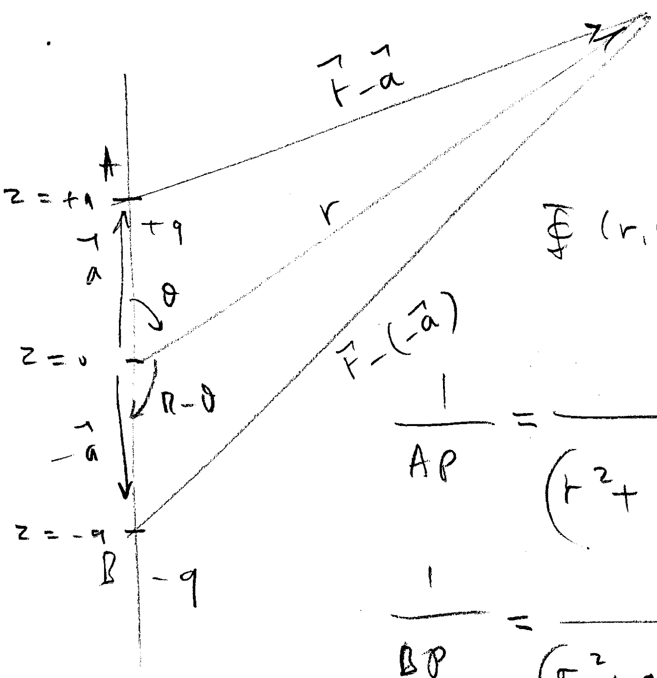
$$\Phi(z) = \frac{2V}{\pi}\frac{R}{z}\sum_\ell \frac{(-1)^\ell}{2\ell+1}\left(\frac{R}{r}\right)^{2\ell} P_{2\ell}(\cos\theta)$$

b) $r < R$ için $\Phi = ?$

burada $\frac{R}{r} \to \frac{r}{R}$ al

c) $C = Q/V = \dfrac{\int \sigma\, da}{V} = R^2/\sinh^{-1}(1)$

3.6.



$$\Phi(r,\theta,\varphi)=\frac{q}{AP}-\frac{q}{BP}$$

$$\frac{1}{AP}=\frac{1}{\left(r^2+a^2-2ar\cos\theta\right)^{1/2}}$$

$$\frac{1}{BP}=\frac{1}{\left(r^2+a^2+2ar\cos\theta\right)^{1/2}}$$

$$\frac{1}{AP}=\frac{1}{\left(r^2+a^2-2ar\cos\theta\right)^{1/2}}=\sum\frac{r_<^{\ell}}{r_>^{\ell+1}}P_\ell(\cos\theta)$$

$$\frac{1}{BP}=\underline{\qquad\qquad}=\sum\frac{r_<^{\ell}}{r_>^{\ell+1}}P_\ell(\cos(\pi-\theta))$$

$$\Phi=q\sum_{\ell=0}^{\infty}\frac{r_<^{\ell}}{r_>^{\ell+1}}\left[P_\ell(\cos\theta)-P_\ell(-\cos\theta)\right]$$

$$=q\begin{cases}0 & \ell \text{ çift} \qquad P_\ell(-x)=(-1)^\ell P_\ell(x)\\ 2q\sum\dfrac{r_<^{\ell}}{r_>^{\ell+1}}P_\ell(\cos\theta) & \ell \text{ tek.}\end{cases}$$

$$\left.\begin{aligned} r>a &\Rightarrow \begin{cases} r_<=a\\ r_>=r\end{cases}\\ r<a &\Rightarrow \begin{cases} r_<=r\\ r_>=a\end{cases}\end{aligned}\right\}\ \varphi\text{'ye bağımlılık yok}$$

BS K

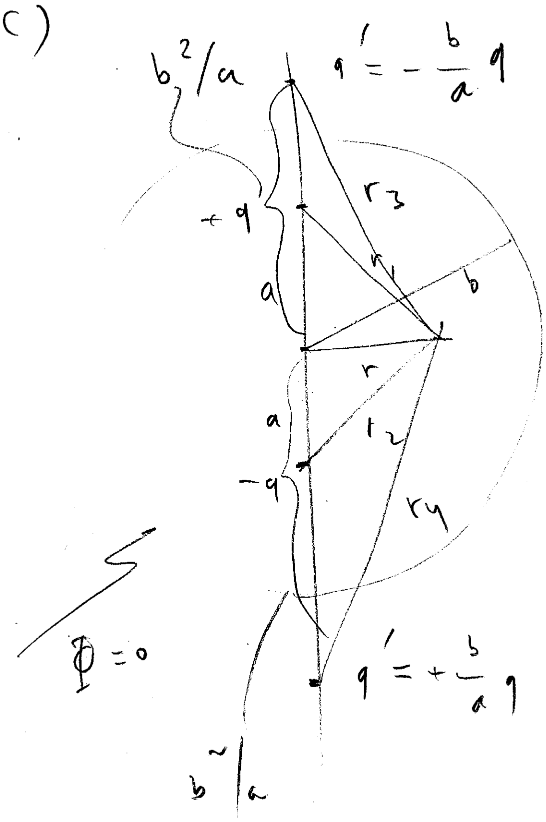
b) $qa = p/2$ alıyoruz.

$$\Phi(r,\theta,\varphi) = \frac{q}{r}\left(1+\frac{a^2}{r^2}-2\frac{a}{r}\cos\theta\right)^{-1/2} - \frac{q}{r}\left(1+\frac{a^2}{r^2}+2\frac{a}{r}\cos\theta\right)^{-1/2}$$

$$r \gg a \Rightarrow \simeq \frac{q}{r}\left(\cancel{1}-\cancel{\frac{a^2}{2r^2}}+\frac{a\cos\theta}{r}+\dots\right)-\frac{q}{r}\left(\cancel{1}-\cancel{\frac{a^2}{2r^2}}-\frac{a\cos\theta}{r}+\dots\right)$$

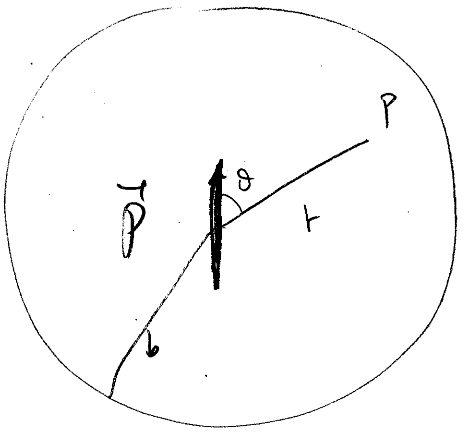
$$\simeq \frac{q}{r}\,\frac{2a\cos\theta}{r} = \frac{P\cos\theta}{r^2} \quad \text{dip. pt.i}$$

c)



$$\Phi_p \simeq \frac{q}{r_1} - \frac{q}{r_2} - \frac{\frac{b}{a}q}{r_3} + \frac{\frac{b}{a}q}{r_4}$$

Not: sonsuz q yükü var.

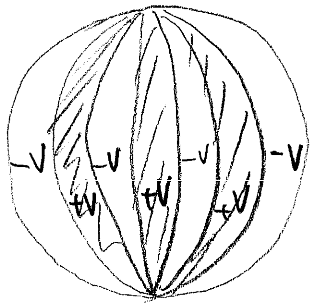


$$\Phi_p = \Phi(\text{dipol katkısı}) + \Phi_p(\text{topraklanmış küre})$$

$$= \frac{P\cos\theta}{r^2} + \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

Sınır koşulu $r = b$ $\Phi = 0$

3.4.)



$$\Phi = \sum_{l} \sum_{m} \left[A_{lm} r^{l} + B_{lm} r^{-(l+1)} \right] Y_{lm}(\theta,\varphi)$$

içeride $B_{lm} = 0$

$$V(\varphi) = \begin{cases} V & 0 < \varphi < \frac{\pi}{n} \\ -V & \frac{\pi}{n} < \varphi < \frac{2\pi}{n} \end{cases}$$

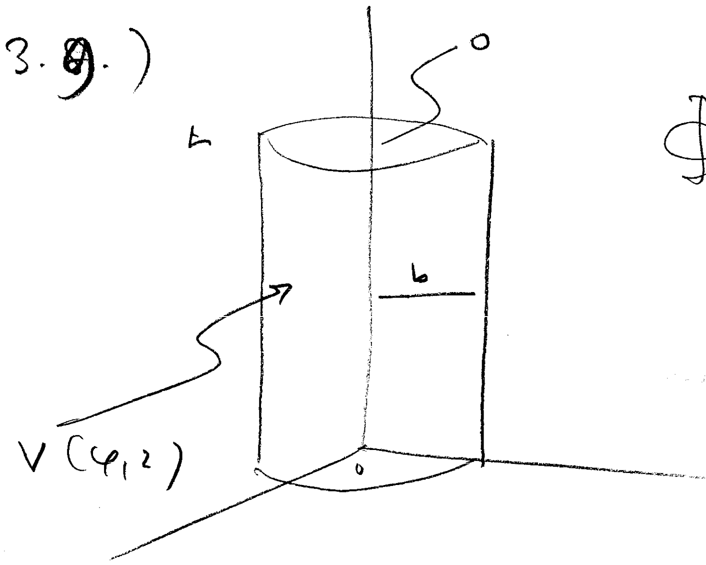
$$V(\varphi) = \sum_{l} \sum_{m} A_{lm} a^{l} Y_{lm}(\theta,\varphi)$$

$$\iint_{\Omega} V(\varphi)\, Y_{l'm'}(\theta,\varphi)\, d\Omega = \sum_{l} \sum_{m} A_{lm} a^{l} \iint_{\Omega} Y_{lm}(\theta,\varphi) Y_{l'm'}(\theta,\varphi')\, d\Omega$$

$$= A_{lm} a^{l}$$

$$A_{lm} = \frac{C}{a^{l}} \int_{0}^{\pi} \sin\theta\, d\theta\, V(\varphi) P_{l}^{m}(\cos\theta)\, \delta_{m0}$$

3.9.)



$$\Phi(\rho,\varphi,z) = R(\rho)Q(\varphi)Z(z)$$

$$Q(\varphi) = A\sin n\varphi + B\cos n\varphi$$

$$Z(z) = \sin k_n z$$

$$R(\rho) = C J_m(k_n\rho) + D N_m(k_n\rho)$$

$$z(0) = 0 = z(L)$$

$\sin k_n L = n\pi \Rightarrow k_n = \frac{n\pi}{L}$ iç bölgede $D \to 0$

N'lerin ρ'lar negatif kuvvete sahip.

$$\Phi(\rho,\varphi,z) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} J_m(k_n\rho)\sin\frac{n\pi z}{L}\left(A_{nm}\sin m\varphi + B_{nm}\cos m\varphi\right)$$

$$A_{mn} = \frac{2}{\pi L J_m(k_n b)}\int_0^{2\pi}\int_0^{L} d\varphi\, z\, dz \sin m\varphi \sin\frac{n\pi z}{L} V(\varphi,z)$$

$$B_{mn} = \frac{2}{\pi L J_m(k_n b)}\int_0^{2\pi}\int_0^{L} d\varphi\, z\, dz \cos m\varphi \sin\frac{n\pi z}{L} V(\varphi,z)$$

3.9. $V(\varphi,z) = \begin{cases} +V & -\frac{\pi}{2} < \varphi < \frac{\pi}{2} \\ -V & \pi/2 < \varphi < 3\pi/2 \end{cases}$ $A_{mn} = 0$

$$B_{mn} = \frac{8V}{nm\pi^2 J_m(k_n b)}\sin\frac{m\pi}{2}\left(\begin{matrix} m \text{ tek} \\ n \text{ tek.}\end{matrix}\right.$$