BÖLÜM 2

DALGA PAKETLERİ ve BELİRSİZLİK BAĞINTILARI

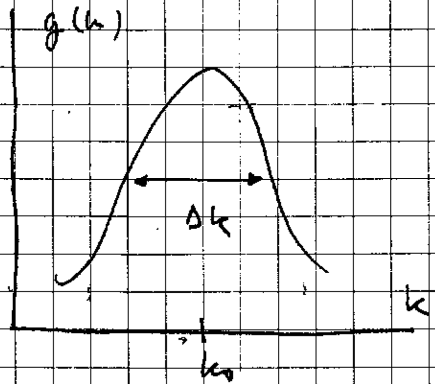
Davranışları dalga gibi olan parçacıkların şekillenimlerini düşünmek zor olduğundan dolayı, uzayın belirli bir bölgesine lokalize olmuş dalgalar üzerine yoğunlaşırız. Dalgaları bu şekilde lokalize etmek için Fourier integrallerinden yararlanırız:

$$f(x) = \int_{-\infty}^{+\infty} dk\, \underbrace{g(k)}_{\text{ağırlık fonk.}}\, e^{ikx}$$

(g(k)'yı uygun bir şekilde seçerek uzayın belirli bir bölgesinde sıfırdan farklı veya küçük yapabiliriz!)

bu $\lambda = 2\pi/k$ dalga boyundaki dalgaların bir lineer süperpozisyonu'dur.

$$g(k) = e^{-\alpha(k-k_0)^2} \qquad \text{Gauss } \cancel{\text{dağ.}}$$



$$\Rightarrow f(x) = \int_{-\infty}^{+\infty} dk\, e^{-\alpha(k-k_0)^2}\, e^{ikx}$$

$$= \int_{-\infty}^{+\infty} dk\, e^{-\alpha(k-k_0)^2 + i(k-k_0)x + ik_0x}$$

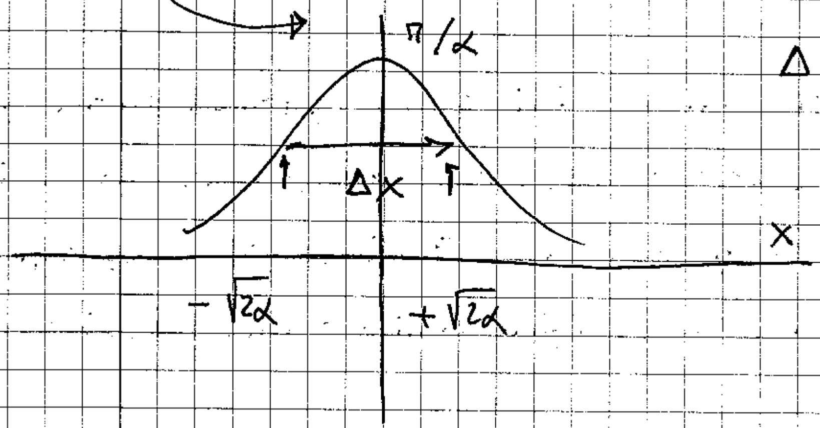
$$= e^{ik_0x} \int_{-\infty}^{+\infty} dk'\, e^{-\alpha k'^2 + ik'x}$$

$$= e^{ik_0x} \int_{-\infty}^{+\infty} dk'\, e^{-\alpha\left(k' + \frac{ix}{2\alpha}\right)^2 - \frac{x^2}{4\alpha}}$$

$$= e^{ik_0x}\, e^{-x^2/4\alpha} \underbrace{\int_{-\infty}^{+\infty} dk''\, e^{-\alpha k''^2}}_{\sqrt{\pi/\alpha}} = \sqrt{\frac{\pi}{\alpha}}\, \underbrace{e^{ik_0x}}_{\text{faz çarpanı}}\, e^{-x^2/4\alpha}$$

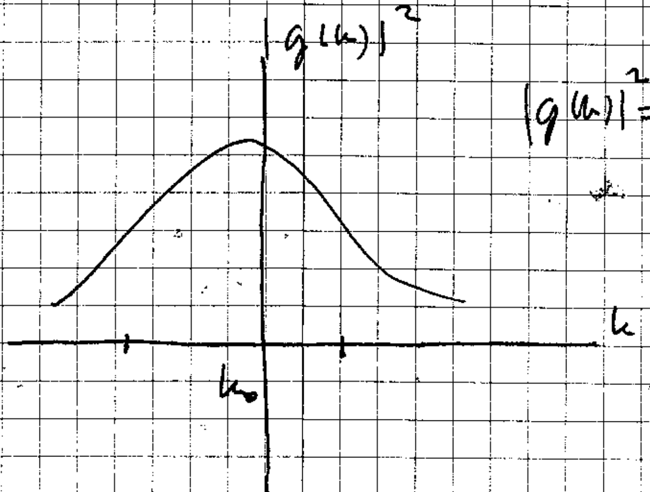
BS K

$$|f(x)|^2 = \frac{\pi}{\alpha}\, e^{-x^2/2\alpha}$$



$\Delta x = 2\sqrt{2\alpha}$

$x = \pm\sqrt{2\alpha}$, $|f(x)|^2$ tepe değerinin (ki bu $\frac{\pi}{\alpha}$) $1/e$'sine düşer.



$|g(k)|^2 = e^{-2\alpha(k-k_0)^2}$ $\qquad -2\alpha(k-k_0)^2 = 1$

$\Rightarrow\ k - k_0 = \mp \frac{1}{\sqrt{2\alpha}}$

$\Delta k = \frac{2}{\sqrt{2\alpha}}$

α küçük seçilirse $\Rightarrow$ $|f(x)|^2$ x-uzayında lokalize ama
k- " genişliyor.

ya da tam tersi !!.

$\Delta k\, \Delta x = \frac{2}{\sqrt{2\alpha}} \cdot 2\sqrt{2\alpha} = 4$ α'dan bağımsız.

İkisini birden küçük yapmak olanaksızdır.

$f(x)$ zaman içinde nasıl yayılır? Tek tek dalgaların nasıl yayıldığını bilmeliyiz.

e^{ikx} : basit düzlem dalga

$$\longrightarrow \quad e^{ikx - i\omega t} = e^{i(kx - \omega t)} \qquad \omega = 2\pi\nu = 2\pi \frac{c}{\lambda}$$

$$= e^{ik(x - ct)} \qquad k = \frac{2\pi}{\lambda} \Big\} = ck$$

$$f(x,t) = \int_{-\infty}^{+\infty} dk\, g(k)\, e^{ik(x-ct)}$$

$$= f(x - ct)$$

$x = 0$ yerine $x = ct$'ye lokalize olunur.

EMD paketi şeklini bozmadan c hızı ile yayılır.

$\omega = ck$ dağınımsız ortamlarda

Bir parçacık için $\omega = \omega(k)$

$$\Rightarrow \quad f(x,t) = \int_{-\infty}^{+\infty} dk\, g(k)\, e^{i[kx - \omega(k)t]}$$

integral k_0'da merkezlendiğinden bu civarda $\omega(k)$'yı seriye açarız.

BS K

4

$$\omega(k) \simeq \omega(k_0) + (k-k_0)\overbrace{\left(\frac{d\omega}{dk}\right)_{k_0}}^{v_g} + \frac{1}{2}(k-k_0)^2 \overbrace{\left(\frac{d^2\omega}{dk^2}\right)_{k_0}}^{\beta} + \dots$$

$$f(x,t) = \int_{-\infty}^{+\infty} dk\, e^{-\alpha(k-k_0)^2} e^{ikx - i\omega(k_0)t - iv_g(k-k_0)t - i\beta(k-k_0)^2 t/2}$$

$$= e^{-i\omega(k_0)t + ik_0 x}$$

$$\times \int_{-\infty}^{+\infty} dk\, e^{-\alpha(k-k_0)^2 + i(k-k_0)x - iv_g(k-k_0)t - i\beta(k-k_0)^2 t/2}$$

$$= e^{-i\omega(k_0)t + ik_0 x}$$

$$\times \int_{-\infty}^{+\infty} dk\, e^{-\underbrace{(\alpha + i\beta t/2)}_{A}(k-k_0)^2 + i\underbrace{(x - v_g t)}_{B}(k-k_0)}$$

$$= e^{-i\omega(k_0)t + ik_0 x} \int_{-\infty}^{+\infty} dk'\, e^{-Ak'^2 + Bk'}$$

$$= e^{-i\omega(k_0)t + ik_0 x} \int_{-\infty}^{+\infty} dk'\, e^{-A\left(k' - \frac{B}{2A}\right)^2 + \frac{B^2}{4A}}$$

$$= e^{-i\omega(k_0)t + ik_0 x - \{[x - v_g t]^2 / 4(\alpha + i\beta t/2)\}}$$

$$v_g = \left(\frac{d\omega}{dk}\right)_0$$

$$\times \int_{-\infty}^{+\infty} dk''\, e^{-Ak''^2}$$

$$\beta = \left(\frac{d^2\omega}{dk^2}\right)_0$$

$$= \left[\frac{\pi}{\alpha + i\beta t/2}\right]^{1/2} e^{i(k_0 x - \omega(k_0)t)}\, e^{-[(x - v_g t)^2 / 4(\alpha + i\beta t/2)]}$$

BS K

$$|f(x,t)|^2 = \left[\frac{\pi^2}{\alpha^2+\beta^2t^2}\right]^{1/2} e^{-[\alpha(x-v_g t)^2/2(\alpha^2+\beta^2t^2)]}$$

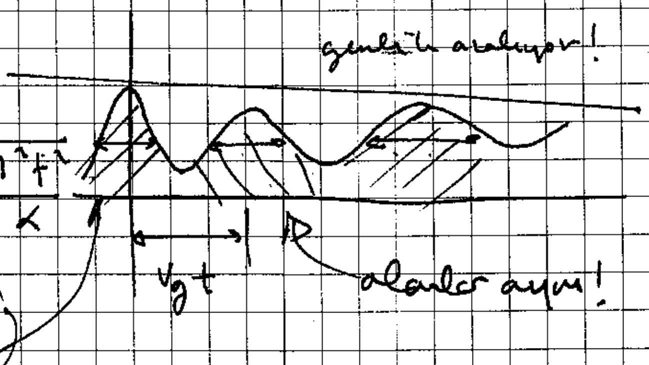
$$\beta = \frac{\beta}{2}$$

tepesi v_g hızı ile giden fakat $t=0$'da genişliği α iken $t=t$'de $\alpha + (\beta^2t^2/\alpha)$ olan bir paket (açılan). (parçacık açılamaz!!) α ne kadar büyükse açılma oranı o kadar küçük olacaktır.

$$|f(x,t)|^2 = \frac{\pi}{\sqrt{\alpha^2+\beta^2t^2}}\, e^{-(x-v_g t)^2/2\left[\frac{\alpha^2+\beta^2t^2}{\alpha}\right]}$$

$$\Delta x = 2\,\frac{\sqrt{2(\alpha^2+\beta^2t^2)}}{\alpha} = 2\sqrt{2}\,\sqrt{\alpha+\frac{\beta^2t^2}{\alpha}}$$

$t=0$ da $2\sqrt{2\alpha}$



$f(x,t)$ parçacığı temsil ediyor $\Rightarrow$ $v_g = \frac{d\omega}{dk} = \frac{p}{m}$

$E = \hbar\omega = \frac{p^2}{2m} \Rightarrow \omega = \frac{p^2}{2m\hbar} \Rightarrow p = \hbar k$ olmalı. $\left[k = \frac{p}{\hbar},\ \omega = \frac{E}{\hbar}\right]$

$$f(x,t) = \int_{-\infty}^{+\infty} \frac{dp}{\hbar}\, g(p)\, e^{i(px-Et)/\hbar}$$ ya da

$$\boxed{\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int dp\, \phi(p)\, e^{(i/\hbar)\left(px-\frac{p^2}{2m}t\right)}}$$

BS K

$$\frac{\partial \psi(x,t)}{\partial t} = \frac{1}{\sqrt{2\pi\hbar}} \int dp\, \phi(p) \frac{-i}{\hbar} E\, e^{i(px-Et)/\hbar}$$

$$= \frac{-i}{\hbar} \frac{1}{\sqrt{2\pi\hbar}} \int dp\, \phi(p) \frac{p^2}{2m} e^{i(px-Et)/\hbar}$$

$$\Rightarrow \quad i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}$$ Schrödinger Denk.

Belirsizlik Bağıntıları:

$\Delta k\, \Delta x \gtrsim 1$ ya da $\Delta p\, \Delta x \geq \hbar$ $\quad p = \hbar k$

$$\frac{p}{m} \Delta p \frac{m}{p} \Delta x \geq h$$

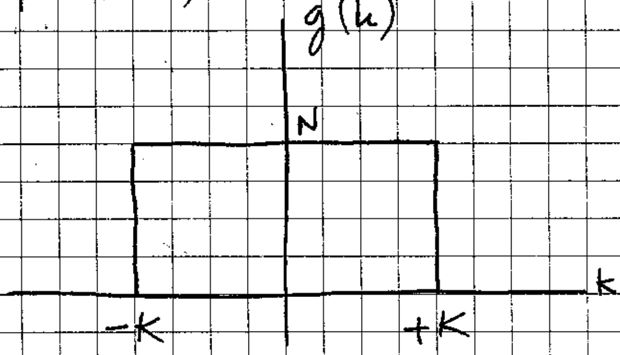
$$\Leftarrow \begin{cases} \Delta E = \dfrac{p \Delta p}{m} \\ p = m \dfrac{\Delta x}{\Delta t} \end{cases}$$

$\Delta E\, \Delta t \geq$!

BS K

BÖLÜM 2 (Problemler)

prb.2.1.)



$\Delta K = 2K$

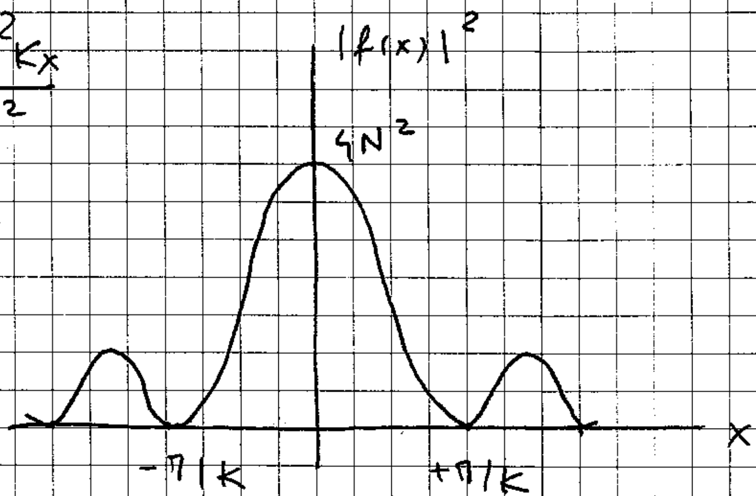
(a) $$f(x) = \int_{-\infty}^{+\infty} dk\, g(k)\, e^{ikx} = N\int_{-K}^{+K} dk\, e^{ikx} = 2N\,\frac{\sin Kx}{x}$$

(b) $$\int_{-\infty}^{+\infty} dx\, |f(x)|^2 = 8N^2 \int_0^{\infty} dx\, \frac{\sin^2 Kx}{x^2} = 8KN^2 \underbrace{\int_0^{\infty} dy\, \frac{\sin^2 y}{y^2}}_{\pi/2}$$

$$\Rightarrow\ 4KN^2 = 1\ ,\quad N = \frac{1}{\sqrt{4\pi K}}$$

(c) $$\int_{-\infty}^{+\infty} dk\, |g(k)|^2 = N^2 \int_{-K}^{+K} dk = 2KN^2 = \frac{1}{2\pi}\ ;\quad N = \frac{1}{\sqrt{4\pi K}}$$

(d) $$|f(x)|^2 = 4N^2\,\frac{\sin^2 Kx}{x^2}$$



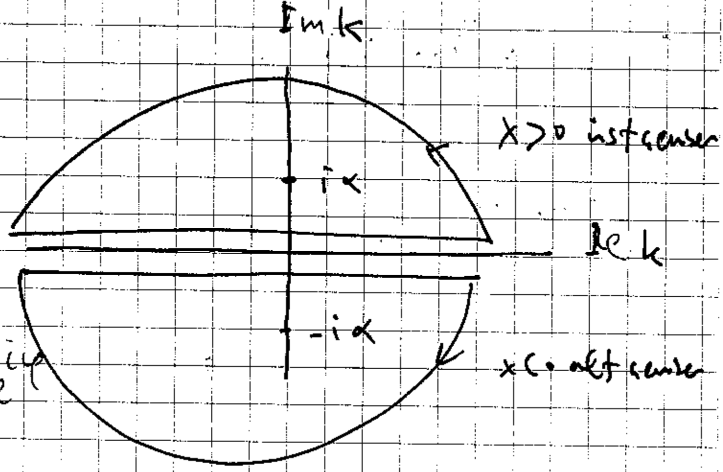
$\Delta x \sim \frac{2\pi}{K}$

$\Delta K \sim 2K$

$\Delta k\, \Delta x \sim 4\pi > 1$

BS K

prb. 2.2.) $g(k) = \frac{N}{k^2+\alpha^2}$



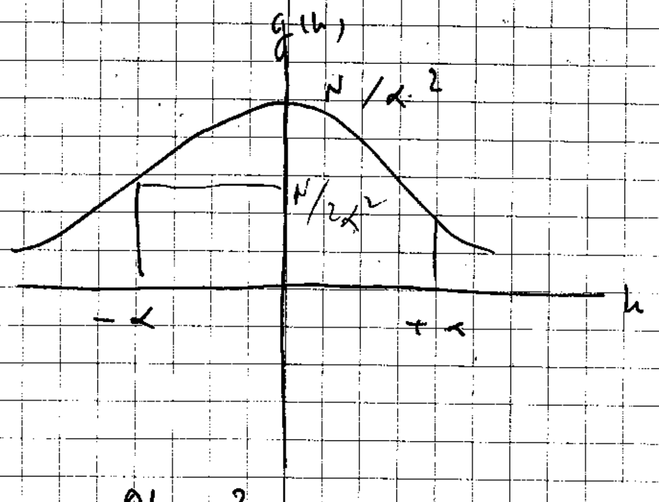
$$f(x) = N\int_{-\infty}^{+\infty} dk \, \frac{e^{ikx}}{k^2+\alpha^2}$$

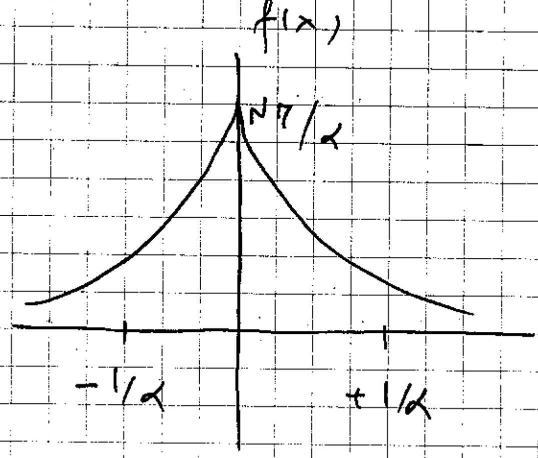
$$\Rightarrow \frac{1}{N} f(x) = \int dk \, \frac{e^{ikx}}{(k-i\alpha)(k+i\alpha)}$$

$k=i\alpha$ $x>0$ için; $\frac{1}{N} f(x) = 2\pi i \, \mathrm{Res}\left[\frac{e^{ikx}}{(k-i\alpha)(k+i\alpha)} \times (k-i\alpha)\right] = 2\pi i \, \frac{e^{-\alpha x}}{2i\alpha} = \frac{\pi}{\alpha} e^{-\alpha x}$

$k=-i\alpha$ $x<0$ 〃 ; $\frac{1}{N} f(x) = -2\pi i \, \mathrm{Res}\left[\frac{e^{ikx}}{(k-i\alpha)(k+i\alpha)} \times (k+i\alpha)\right] = -2\pi i \, \frac{e^{\alpha x}}{-2i\alpha} = \frac{\pi}{\alpha} e^{\alpha x}$

$$f(x) = N\frac{\pi}{\alpha} e^{-\alpha|x|}$$





$\Delta k \sim 2\alpha$

$\Delta x \sim \frac{2}{\alpha}$

$\Delta k \Delta x \simeq 4$

BS K

prb. 2.3.) $\omega(k) = \frac{\hbar k^2}{2m}$

$$(2.17.)\quad |f(x,t)|^2 = \frac{\pi}{\sqrt{\alpha^2+\beta^2t^2}} \exp\left\{ -\frac{\alpha(x-v_g t)^2}{2(\alpha^2+\beta^2t^2)} \right\}$$

$$\Delta x = 2\left[\frac{2(\alpha^2+\beta^2t^2)}{\alpha}\right]^{1/2} = 2\sqrt{2\alpha + 2\frac{\beta^2t^2}{\alpha}}$$

a) $t=0$ da $\Delta x_0 = 2\sqrt{2\alpha}$, $t=1$'de $\Delta x_1 = 2\sqrt{2\alpha + \frac{2\beta^2}{\alpha}}$

$$= 2\sqrt{2\alpha}\sqrt{1+(\beta/\alpha)^2}$$

kesirsel değişim: $\frac{\Delta x_1 - \Delta x_0}{\Delta x_0} = \frac{\Delta x_1}{\Delta x_0} - 1$

$$\beta = \frac{1}{2}\frac{d^2\omega}{dk^2} = \frac{1}{2}\frac{d}{dk}\left(\frac{\hbar k}{m}\right) = \frac{\hbar}{2m}$$

$\Delta x_0 = 10^{-4}$ cm , $\alpha = \frac{(\Delta x_0)^2}{8} = \frac{1}{8}.10^{-8}$ cm^2 $\quad \beta = 0.6$ cm^2/sn

$$\frac{\Delta x_1 - \Delta x_0}{\Delta x_0} = \frac{\cancel{\Delta x_0}\sqrt{1+(\beta/\alpha)^2}}{\cancel{\Delta x_0}} - 1 \simeq \frac{\beta}{\alpha} = 4.8\times 10^{8}$$

$$\Delta x_0 = 10^{-8}\text{ cm} \Rightarrow \frac{\Delta x_1 - \Delta x_0}{\Delta x_0} = \frac{\beta}{\alpha} = 4.8\times 10^{16}$$

b) $m = 1$ gr $\quad \Delta x_0 = 1$ cm $\Rightarrow \alpha = \frac{1}{8}$ cm^2 $\quad \beta \simeq \frac{1}{2}\times 10^{-27}$ cm^2/sn

$$\frac{\Delta x_1}{\Delta x_0} - 1 = \sqrt{1+(\beta/\alpha)^2} - 1 \simeq \cancel{1} + \frac{1}{2}\left(\frac{\beta}{\alpha}\right)^2 - \cancel{1} \simeq 16\times 10^{-54} \quad !!!$$

Makroskopik cisim dağılmaz

BS K

prb.2.4.)

$$E_k + mc^2 = \gamma mc^2 = \frac{mc^2}{\sqrt{1-\beta^2}} \Rightarrow \beta = \frac{\sqrt{E_k^2 + 2E_k mc^2}}{E_k + mc^2}$$

$$E_k = 13.6\,eV \Rightarrow v \simeq \sqrt{\frac{2E_k}{mc^2}}\,c \simeq \sqrt{\frac{27.2}{510.000}}\,c = 0.0073c \Rightarrow t = \frac{10^4}{0.0073c} = 4.57s$$

$$E_k = 100\,MeV \Rightarrow \beta \simeq 1 \quad v = 3.10^5\ km/s \quad t = \frac{10^4}{3.10^5} = \frac{1}{30}\,s$$

$t=0$ da $\Delta x_0 = 2\sqrt{2\alpha} = 10^{-1}\,cm$ $\quad$ t anında $\Delta x_t = \Delta x_0 \sqrt{1+\beta^2 t^2/\alpha^2}$

a) $t = 4.57\,sn \quad \beta = 0.6\,cm^2/sn \quad \alpha = \frac{(\Delta x_0)^2}{8} = \frac{10^{-2}}{8}\,cm^2$

$$\Delta x = 10^{-1}\sqrt{1 + \frac{0.36 \times 20.9 \times 64}{10^{-4}}} \simeq 220\,cm$$

b) $t = \frac{1}{30}\,sn \Rightarrow \Delta x \simeq 10^{-1}\sqrt{1 + \frac{0.36 \times 64}{900 \times 10^{-4}}} \simeq 1.6\,cm$

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prb 2.5.)

$$\lambda = \frac{c}{\sqrt{\nu^2 - \nu_0^2}} \Rightarrow \nu^2 - \nu_0^2 = \frac{c^2}{\lambda^2}$$

$$\Rightarrow \omega^2 - \omega_0^2 = k^2 c^2$$

$$\Rightarrow \omega = \sqrt{\omega_0^2 + k^2 c^2}$$

$$v_g = \left.\frac{d\omega}{dk}\right|_{k=k_0} = \frac{1}{2}\left(\omega_0^2 + k^2 c^2\right)^{-1/2} 2.k\,c^2 \Big|_{k_0} = \frac{k_0 c^2}{\sqrt{\omega_0^2 + k_0^2 c^2}}$$

$$\omega_0 \to 0 \Rightarrow v_g \to c$$

BS K

prb. 2.6.) $\nu = \sqrt{\dfrac{2\pi T}{\rho \lambda^3}}$ T : yüzey gerilimi

ρ : yoğunluk $\lambda = 2\pi / k$

$$\frac{\omega}{2\pi} = \left[\frac{2\pi T}{\rho} \left(\frac{k}{2\pi} \right)^3 \right]^{1/2} \Rightarrow \omega = \sqrt{\frac{T}{\rho}}\, k^{3/2}$$

$$v_g = \frac{d\omega}{dk} = \frac{3}{2} \sqrt{\frac{T}{\rho}}\, k^{1/2} = \frac{3}{2} \sqrt{\frac{kT}{\rho}}$$

$$v_p = \lambda \nu = \sqrt{\frac{2\pi T}{\rho \lambda}} = \sqrt{\frac{kT}{\rho}}$$

$$v_g = \frac{3}{2} v_p$$

$$\nu = \sqrt{\frac{g}{2\pi\lambda}}, \quad \omega = \sqrt{gk} \qquad v_g = \frac{1}{2} \left(g/k \right)^{1/2}$$

$$v_p = \lambda \nu = \left(g/k \right)^{1/2} \qquad v_g = 1/2\, v_p$$

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prb. 2.7.) $E = \dfrac{p^2}{2m} + \dfrac{1}{2m} \omega^2 x^2$

$\Delta x \Delta p \sim \hbar \Rightarrow x p \sim \hbar$

$x \sim \hbar / p$

$x \sim a \Rightarrow p \sim \hbar / 2a$

genlik

$$E = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 \frac{\hbar^2}{p^2}$$

$$\frac{dE}{dp} = 0 \Rightarrow \frac{p}{m} - \frac{1}{\cancel{2}} m \omega^2 \hbar^2 \frac{\cancel{2}}{p^3} \Rightarrow p^4 - m^2 \omega^2 \hbar^2 = 0 \Rightarrow p^2 = m\omega\hbar$$

$$E_{min} = \frac{\cancel{m}\omega\hbar}{2\cancel{m}} + \frac{1}{2} \cancel{m} \omega^{\cancel{2}} \hbar^{\cancel{2}} \frac{1}{\cancel{m\omega\hbar}} = \hbar\omega$$

BS K

prb.2.8.) $\frac{1}{\tau} = \frac{1}{144} \frac{me^{10}}{c^3 \hbar^6}$ $\Delta E\, \tau \sim \hbar$

$$\Delta E = \frac{1}{144} \frac{me^{10}}{c^3 \hbar^5} = \frac{1}{144} mc^2 \alpha^5$$

$$E_2 - E_1 = \frac{1}{2} mc^2 \alpha^2 \left(1 - \frac{1}{4}\right) = \frac{3}{8} mc^2 \alpha^2$$

$$\frac{\Delta E}{E_2 - E_1} \sim \alpha^2 = 10^{-4}$$

BS K