BÖLÜM 3

SCHRÖDINGER DALGA DENKLEMİ

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2}$$

En genel çözümü:

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int dp\, \phi(p)\, e^{i\left[px - \frac{p^2}{2m}t\right]/\hbar}$$

$\psi(x,0)$ başlangıç değeri verildiğinde $\psi(x,t)$'lerin bulunabileceği aşikar.

$$\psi(x,0) = \frac{1}{\sqrt{2\pi\hbar}} \int dp\, \phi(p)\, e^{ipx/\hbar}$$

$\psi(x,t)$ için şunları akılda tutmalıyız.

1°) kompleks bir fonksiyondur.

2°) $|\psi(x,t)|$ parçacığın bulunduğunu düşündüğümüz yerde büyük, bunun dışında küçüktür.

$\mathcal{P}(x,t)\, dx = |\psi(x,t)|^2 dx$: $x \mapsto x+dx$ arasında parçacığın t anında bulunma olasılığı (Max Born)

reel

$$\int_{-\infty}^{+\infty} \mathcal{P}(x,t)\,dx = 1$$

koşulu $\psi(x,t)$ leri karelerı integre edilebilen fonksiyonlar sınıfına sokar. : $\int_{-\infty}^{+\infty} |\psi(x,0)|^2 dx < \infty$.

(#) Sonsuz bir integral alındığında $\psi(x,0)$ $x^{-\frac{1}{2}-\epsilon}$, $\epsilon > 0$ hızı ile sıfıra gitmelidir.

(#) $\psi(x,t)$: x'e göre sürekli olmalı.

$\psi_1(x,t)$ ve $\psi_2(x,t)$ S.D. birer çözümleri

$\Rightarrow \psi(x,t) = \alpha_1 \psi_1 + \alpha_2 \psi_2 \qquad \alpha_1, \alpha_2 \in \mathbb{C}$

de bir çözümdür.

BS K

Şimdi parçacığın uzayın bir bölgesinde bulunma olasılığının zamanla değişimine bakalım:

$$\frac{\partial}{\partial t}\mathcal{P}(x,t) = \frac{\partial}{\partial t}\psi^*(x,t)\,\psi(x,t)$$

$$= \psi\frac{\partial \psi^*}{\partial t} + \psi^*\frac{\partial \psi}{\partial t}$$

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} \qquad /\psi^*$$

$$-i\hbar\frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi^*}{\partial x^2} \qquad /\psi$$

$$\frac{\partial}{\partial t}\mathcal{P}(x,t) = \frac{1}{i\hbar}\left\{-\frac{\hbar^2}{2m}\psi^*\frac{\partial^2 \psi}{\partial x^2} + \frac{\hbar^2}{2m}\psi\frac{\partial^2 \psi^*}{\partial x^2}\right\}$$

$$= \frac{\hbar}{2im}\left\{-\psi^*\frac{\partial^2 \psi}{\partial x^2} + \psi\frac{\partial^2 \psi^*}{\partial x^2}\right\}$$

$$= \frac{\hbar}{2im}\left\{\frac{\partial}{\partial x}\left[\frac{\partial \psi^*}{\partial x}\psi\right] - \cancel{\frac{\partial \psi^*}{\partial x}\frac{\partial \psi}{\partial x}}\right.$$

$$\left. - \frac{\partial}{\partial x}\left[\frac{\partial \psi}{\partial x}\psi^*\right] + \cancel{\frac{\partial \psi^*}{\partial x}\frac{\partial \psi}{\partial x}}\right\}$$

$$= -\frac{\partial}{\partial x}\left\{\frac{\hbar}{2im}\left[\psi^*\frac{\partial \psi}{\partial x} - \psi\frac{\partial \psi^*}{\partial x}\right]\right\}$$

$$= -\frac{\partial}{\partial x}\, j(x,t)$$

$$j(x,t) = \frac{\hbar}{2im}\left[\psi^*\frac{\partial \psi}{\partial x} - \psi\frac{\partial \psi^*}{\partial x}\right]$$

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$$\frac{\partial}{\partial t}\rho(x,t)+\frac{\partial}{\partial x}j(x,t)=0.$$ korunum yas.

bir bölgede olasılık yoğunluğunda zamanla meydana gelen bir değişme o bölgeye giren akıdaki yerel bir değişim ile karşılanır demektedir.

$$\int_{-\infty}^{+\infty}\rho(x,t)\,dx=1.$$

karesi integre edilebilen bir fonk. ∞'da sıfır

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty}\rho(x,t)\,dx=-\int_{-\infty}^{+\infty}dx\,\frac{\partial}{\partial x}j(x,t)$$

$$=-j(x,t)\Big|_{-\infty}^{+\infty}=0$$

(a,b) aralığında olsa idi

$$\frac{\partial}{\partial t}\int_{a}^{b}\rho(x,t)\,dx=j(a,t)-j(b,t)$$

(#) ~~Ker...~~ Bir pot. etkisinde hareket eden parçacığın SDD.

$$i\hbar\frac{\partial\psi(x,t)}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}+V(x)\psi(x,t)$$

reel.

korunum yasası $V(x)$ reel $\neq$ geçerlidir.

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$$\frac{\partial \rho}{\partial t} = \psi \frac{\partial \psi^*}{\partial t} + \psi^* \frac{\partial \psi}{\partial t}$$

$$\begin{cases} \dfrac{\partial \psi}{\partial t} = -\dfrac{\hbar}{2im}\dfrac{\partial^2 \psi}{\partial x^2} + V(x)\dfrac{1}{i\hbar}\psi & / \psi^* \\ \dfrac{\partial \psi^*}{\partial t} = \dfrac{\hbar}{2im}\dfrac{\partial^2 \psi^*}{\partial x^2} - V^*(x)\dfrac{1}{i\hbar}\psi^* & / \psi \end{cases}$$

$$\frac{\partial \rho}{\partial t} = \frac{\hbar}{2im}\psi\frac{\partial^2 \psi^*}{\partial x^2} - \cancel{\frac{1}{i\hbar}\psi\psi^* V^*}$$

$$- \frac{\hbar}{2im}\psi^*\frac{\partial^2 \psi}{\partial x^2} + \cancel{\frac{1}{i\hbar}\psi\psi^* V}$$

V : reel.

$$= -\frac{\partial}{\partial x} j(x,t) \quad \left(+ \frac{1}{i\hbar}|\psi|^2 \left[V - V^* \right] \right)$$

(4) Tüm yaptıklarımızı 3-boyuta genelleştiriniz.

1°) $$i\hbar \frac{\partial \psi(\vec{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\vec{\nabla}^2 \psi(\vec{r},t) + V(\vec{r})\,\psi(\vec{r},t)$$

2°) $$\frac{\partial}{\partial t}\rho(\vec{r},t) + \vec{\nabla}\cdot\vec{j}(\vec{r},t) = 0$$

$$\rho(\vec{r},t) = |\psi(\vec{r},t)|^2$$

3°) $$\vec{j}(\vec{r},t) = \frac{\hbar}{2im}\left[\psi^*(\vec{r},t)\vec{\nabla}\psi(\vec{r},t) - \psi(\vec{r},t)\vec{\nabla}\psi^*(\vec{r},t)\right]$$

Beklenen Değer: $\rho(x,t)$'ler verildiğinde x'in fonksiyonlarının beklenen değerleri hesaplanabilir;

$$\langle f(x)\rangle = \int dx\, f(x)\,\rho(x,t)$$

Klasik olarak, $p = mv = m\dfrac{dx}{dt}$

$$\langle p\rangle = m\frac{d}{dt}\langle x\rangle = m\frac{d}{dt}\int dx\, \psi^*(x,t)\, x\, \psi(x,t)$$

$$= m\int_{-\infty}^{+\infty} dx\left(\frac{\partial\psi^*}{\partial t}x\psi + \psi^* x\frac{\partial\psi}{\partial t}\right)$$

x açık olarak t'ye bağlı değil.

$$= \frac{\hbar}{2i}\int_{-\infty}^{+\infty} dx\left(\frac{\partial^2\psi^*}{\partial x^2}x\psi - \psi^* x\frac{\partial^2\psi}{\partial x^2}\right)$$

(*)

$$\frac{\partial^2\psi^*}{\partial x^2}x\psi = \frac{\partial}{\partial x}\left(\frac{\partial\psi^*}{\partial x}x\psi\right) - \frac{\partial\psi^*}{\partial x}\frac{\partial}{\partial x}(x\psi)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\psi^*}{\partial x}x\psi\right) - \underbrace{\psi\frac{\partial\psi^*}{\partial x}} - x\frac{\partial\psi^*}{\partial x}\frac{\partial\psi}{\partial x}$$

$$\# \quad \frac{\partial}{\partial x}\psi^*\psi = \psi\frac{\partial\psi^*}{\partial x} + \psi^*\frac{\partial\psi}{\partial x}$$

$$= \frac{\partial}{\partial x}\left[\left(\frac{\partial\psi^*}{\partial x}x\psi\right) - \psi^*\psi\right] + \psi^*\frac{\partial\psi}{\partial x} - x\frac{\partial\psi^*}{\partial x}\frac{\partial\psi}{\partial x}$$

$$\frac{\partial^2\psi}{\partial x^2}x\psi^* = \frac{\partial}{\partial x}\left[\left(\frac{\partial\psi}{\partial x}x\psi^*\right) - \psi^*\psi\right] + \psi\frac{\partial\psi^*}{\partial x} - x\frac{\partial\psi^*}{\partial x}\frac{\partial\psi}{\partial x}$$

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$$\frac{\partial}{\partial x}\left(\psi^* x \frac{\partial \psi}{\partial x}\right) = \underbrace{x\,\frac{\partial \psi^*}{\partial x}\frac{\partial \psi}{\partial x}} + \psi^*\frac{\partial \psi}{\partial x} + x\psi^*\frac{\partial^2 \psi}{\partial x^2}$$

$$\frac{\partial^2 \psi^*}{\partial x^2}\, x\psi = \frac{\partial}{\partial x}\left[x\psi\frac{\partial \psi^*}{\partial x} - \psi^*\psi\right] + \psi^*\frac{\partial \psi}{\partial x} + \psi^*\frac{\partial \psi}{\partial x} + x\psi^*\frac{\partial^2 \psi}{\partial x^2}$$
$$- \frac{\partial}{\partial x}\left(\psi^* x \frac{\partial \psi}{\partial x}\right)$$

$$\Rightarrow\ \frac{\partial^2 \psi^*}{\partial x^2}\, x\psi - x\psi^*\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x}\left[x\psi\frac{\partial \psi^*}{\partial x} - x\psi^*\frac{\partial \psi}{\partial x}\right]$$
$$- \frac{\partial}{\partial x}|\psi|^2 + 2\psi^*\frac{\partial \psi}{\partial x}$$

$\to 0$ kısmi int. edi. ∞'da sıfır.

$$\langle p\rangle = \frac{\hbar}{2i}\int_{-\infty}^{+\infty} dx \left\{\cancel{\frac{\partial}{\partial x}\left[x\psi\frac{\partial \psi^*}{\partial x} - x\psi^*\frac{\partial \psi}{\partial x} - \psi\psi^*\right]} + 2\psi^*\frac{\partial \psi}{\partial x}\right\}$$

$$= \frac{\hbar}{\cancel{2}i}\cancel{2}\int_{-\infty}^{+\infty} dx\ \psi^*\frac{\partial}{\partial x}\psi$$

$$= \int_{-\infty}^{+\infty} dx\ \psi^*\ \frac{\hbar}{i}\frac{\partial}{\partial x}\psi$$

$$p = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

$$\langle f(p)\rangle = \int_{-\infty}^{+\infty} dx\ \psi^*\ f\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi$$

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$\phi(p)$'nin fiziksel anlamı:

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int dp\, \phi(p)\, e^{ipx/\hbar} = \sqrt{\frac{\hbar}{2\pi}} \int dk\, \phi(\hbar k)\, e^{ikx}$$

$$\begin{cases} \phi(\hbar k) = \dfrac{1}{\sqrt{2\pi\hbar}} \displaystyle\int dx\, \psi(x)\, e^{-ikx} \\[2ex] \phi(p) = \dfrac{1}{\sqrt{2\pi\hbar}} \displaystyle\int dx\, \psi(x)\, e^{-ipx/\hbar} \end{cases}$$

$$\Rightarrow \int dp\, \phi^*(p)\, \phi(p) = \int dp\, \phi^*(p) \frac{1}{\sqrt{2\pi\hbar}} \int dx\, \psi(x)\, e^{-ipx/\hbar}$$

$$= \int dx\, \psi(x) \frac{1}{\sqrt{2\pi\hbar}} \int dp\, \phi^*(p)\, e^{-ipx/\hbar}$$

$$= \int dx\, \psi(x)\, \psi^*(x) = 1 \quad \text{(Parseval th.)}$$

$$\langle p \rangle = \int dx\, \psi^* \frac{\hbar}{i} \frac{d}{dx} \psi(x)$$

$$= \int dx\, \psi^*(x) \frac{\hbar}{i} \frac{d}{dx} \frac{1}{\sqrt{2\pi\hbar}} \int dp\, \phi(p)\, e^{ipx/\hbar}$$

$$= \int dp\, \phi(p)\, p \frac{1}{\sqrt{2\pi\hbar}} \int dx\, \psi^*(x)\, e^{ipx/\hbar}$$

$$= \int dp\, \phi(p)\, p\, \phi^*(p)$$

$\phi(p)$: mom. uzayındaki dalga fonk.'u. / $|\phi(p)|^2$: bir zamanda parçacığı p mom.'unda bulma olasılığı.

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int dp\, \phi(p,t)\, e^{ipx/\hbar}$$

Mom. uzayında

$$\langle f(x) \rangle = \int dp\, \phi^*(p,t)\, f\left(i\hbar \frac{\partial}{\partial p}\right) \phi(p,t)$$

olduğu görülebilir. $x = i\hbar \frac{\partial}{\partial p}$

Operatörler:

1°) Her zaman sıradeğiştirmezler

$$[A,B] = AB - BA$$

$$[p,x]\psi(x,t) = \frac{\hbar}{i}\frac{\partial}{\partial x} x\psi - x\frac{\hbar}{i}\frac{\partial}{\partial x}\psi$$

$$= \frac{\hbar}{i}\psi$$

$$[p,x] = \frac{\hbar}{i}$$

2°) p operatöründe i p'nin beklenen değerinin reel olmasını etkilemez.

$$\langle p\rangle - \langle p\rangle^* = \int dx\, \psi^* \frac{\hbar}{i}\frac{\partial\psi}{\partial x} - \int dx\, \psi\left(-\frac{\hbar}{i}\frac{\partial\psi^*}{\partial x}\right)$$

$$= \frac{\hbar}{i}\int dx \left(\psi^*\frac{\partial\psi}{\partial x} + \psi\frac{\partial\psi^*}{\partial x}\right)$$

$$= \frac{\hbar}{i}\int dx\, \frac{\partial}{\partial x}|\psi|^2 = 0$$

(*) Bazen karesi integre edilemeyen fakat belli periyodik koşulları sağlayan fonk.'lar kullanmak gerekebilir.

$$\psi(x) = \psi(x+L) \qquad 0 \leq x \leq L$$

$\frac{\hbar}{i}\frac{d}{dx}$ yine Hermityen bir işlemcidir.

$$\langle p\rangle - \langle p\rangle^* = \frac{\hbar}{i}\int_0^L dx \frac{\partial}{\partial x}(\psi\psi^*)$$

$$= \frac{\hbar}{i}\left[|\psi(L)|^2 - |\psi(0)|^2 \right] = 0$$

Tüm dalga fonk.'ları için beklenen değeri reel olan bir operatöre hermitien operatör denir. $(p, x, \ldots)$

$$p_{op} = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2}$$

$$\longrightarrow \; i\hbar\frac{\partial \psi(x,t)}{\partial t} = \underbrace{\frac{p_{op}^2}{2m}}_{K.E.}\psi(x,t)$$

$$i\hbar\frac{\partial \psi(x,t)}{\partial t} = \left[\frac{p_{op}^2}{2m} + V(x)\right]\psi(x,t)$$

$$= \underbrace{H}_{\text{enerji operatörü}}\psi(x,t) \qquad H = \frac{p^2}{2m} + V(x) \quad \text{Hermitien.}$$

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Özet

1°) Dalga fonk.'larını zaman bağımlılığı 1. dereceden

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = \mathcal{H}\psi(x,t)$$

kısmi diff. denk. ile verilir. $\mathcal{H} = \frac{p^2}{2m} + V(x)$

2°) Dalga fonk.'ları kareli integrallenebilen fonk.'lar sınıfı.

3°) Parçacığın x noktasında bulunması olasılığı,

$$\mathcal{P}(x,t) = |\psi(x,t)|^2$$

4°) $$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int dp\, \phi(p)\, e^{ipx/\hbar}$$

mom. uzayındaki dalga fonk. $|\phi(p)|^2$

5°) p, x : operatörlerdir ve sıradeğiştirmezler

$$p = \frac{\hbar}{i}\frac{\partial}{\partial x}, \quad x = i\hbar\frac{\partial}{\partial p}$$

$$[p, x] = -i\hbar$$

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Örnek: Normalize dalga fonksiyonu

$$\psi(x) = \begin{cases} 2\alpha\sqrt{\alpha}\, x\, e^{-\alpha x} & x>0 \\ 0 & x<0 \end{cases}$$

a) x'in hangi değeri $P(x)$'i maksimum yapar.

$$\frac{d}{dx} P(x) = \frac{d}{dx} |\psi(x)|^2 = 0$$

$$\frac{d}{dx}\left(x^2 e^{-2\alpha x}\right) = 0 \Rightarrow 2x(1-\alpha x)\, e^{-2\alpha x} = 0$$

$$x = 1/\alpha$$

b) parçacığın x $(0, 1/\alpha)$ arasında parçacığın bulunma olasılığı nedir?

$$P = \int_0^{1/\alpha} dx\, x^2 (4\alpha^3) e^{-2\alpha x} = \frac{1}{2}\int_0^2 dy\, y^2 e^{-y} = 0.32$$

c) $\langle x \rangle$, $\langle x^2 \rangle$

$$\langle x \rangle = \int_0^\infty dx\, x \left(4\alpha^3 x^2 e^{-2\alpha x}\right) = \frac{1}{4\alpha}\int_0^\infty dy\, e^{-y} y^3 = \frac{3!}{4\alpha} = \frac{3}{2\alpha}$$

$$\langle x^2 \rangle = \int_0^\infty dx\, x^2 \left(4\alpha^3 x^2 e^{-2\alpha x}\right) = \frac{4!}{8\alpha^2} = \frac{3}{\alpha^2}$$

d) $\phi(p)$, $\langle p \rangle$, $\langle p^2 \rangle$

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_0^\infty dx\, e^{-ipx/\hbar} (2\alpha\sqrt{\alpha})\, x\, e^{-\alpha x}$$

$$= \sqrt{\frac{4\alpha^3}{2\pi\hbar}}\, \frac{d}{d\alpha} \int_0^\infty dx\, e^{-(\alpha + ip/\hbar)x} = -\sqrt{\frac{4\alpha^3}{2\pi\hbar}}\, \frac{1}{(\alpha + ip/\hbar)^2}$$

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$$\langle p \rangle = \int_{-\infty}^{+\infty} dp \, p \, |\phi(p)|^2$$

$$= \frac{4\alpha^3}{2\pi\hbar} \int_{-\infty}^{+\infty} dp \, \frac{p}{(\alpha^2 + p^2/\hbar^2)} = 0$$

$$\langle p^2 \rangle = \frac{4\alpha^3}{2\pi\hbar} \int_{-\infty}^{+\infty} dp \, \frac{p^2}{(\alpha^2 + p^2/\hbar^2)^2} = \frac{8\alpha^3}{2\pi\hbar} \int_0^{\infty} dp \, \frac{p^2}{(\quad)^2}$$

$$p = \hbar\alpha \tan\theta \Rightarrow \langle p^2 \rangle = \int_0^{\pi/2} d\theta \, \sin^2\theta = \alpha^2\hbar^2$$

BS K

3. BÖLÜM

PROBLEMLER

prb. 3.1.) $$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int dp\, \phi(p)\, e^{\frac{i}{\hbar}[px - p^2 t/2m]} \qquad (3.2)$$

$$\psi(x,0) = \frac{1}{\sqrt{2\pi\hbar}} \int dp\, \phi(p)\, e^{ipx/\hbar} \qquad (3.4)$$

burada $$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int dx'\, \psi(x',0)\, e^{-ipx'/\hbar}$$

$$\Rightarrow \psi(x,t) = \int dx' \underbrace{\frac{1}{2\pi\hbar} \int dp\, e^{i[p(x-x') - p^2 t/2m]/\hbar}}_{K(x,x';t)} \psi(x',0)$$

$$= \int dx'\, K(x,x',t)\, \psi(x',0)$$

$$K(x,x',t) = \frac{1}{2\pi\hbar} \int dp\, e^{i[p(x-x') - p^2 t/2m]/\hbar}$$

$$= \frac{1}{2\pi\hbar} e^{im(x-x')^2/2\hbar t} \times \int d\; e^{-it[p - \frac{m}{t}(x-x')]^2/2m\hbar}$$

$$= \sqrt{\frac{m}{2\pi i\hbar t}}\, e^{im(x-x')^2/2\hbar t}$$

$$K(x,x',0) = \frac{1}{2\pi\hbar} \int dp\, e^{ip(x-x')/\hbar} = \frac{1}{2\pi} \int dk\, e^{ik(x-x')} = \delta(x-x')$$

BS K

prb. 3.3.) $V(x)$ kompleks olsun.

$$\frac{\partial}{\partial t}P = \frac{\hbar}{2im}\left[\frac{\partial^2\psi^*}{\partial x^2}\psi - \psi^*\frac{\partial^2\psi}{\partial x^2}\right] + \frac{i}{\hbar}(V^*-V)\psi\psi^*$$

$$\frac{d}{dt}\int dx\, P = \int dx\,\frac{\partial P}{\partial t} = \frac{\hbar}{2im}\cancelto{0}{\int dx\left[\frac{\partial^2\psi^*}{\partial x^2}\psi - \psi^*\frac{\partial^2\psi}{\partial x^2}\right]}$$

$$+\frac{i}{\hbar}\int dx\,(V^*-V)\psi^*\psi$$

$$=\frac{2}{\hbar}\int dx\,\psi^*\psi\,\underline{\mathrm{Im}\,V}$$

$\int dx\, P(x,t)$ zamanla azalıyor $\Rightarrow$ (yani sönümlenme var)

$\frac{d}{dt}\int dx\, P(x,t)$ eğimi negatif olmalı <0

yani, $\mathrm{Im}\,V(x) < 0$ olmalıdır.

prb.3.4.)
$$\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} + \left(\frac{\mu c}{\hbar}\right)^2 \psi = 0 \quad / \psi^*$$

$$\frac{1}{c^2}\frac{\partial^2 \psi^*}{\partial t^2} - \frac{\partial^2 \psi^*}{\partial x^2} + \left(\frac{\mu c}{\hbar}\right)^2 \psi^* = 0 \quad / \psi$$

$-$

$$\frac{1}{c^2}\frac{\partial}{\partial t}\left[\psi^*\frac{\partial \psi}{\partial t} - \psi\frac{\partial \psi^*}{\partial t}\right] = \frac{\partial}{\partial x}\left[\psi^*\frac{\partial \psi}{\partial x} - \psi\frac{\partial \psi^*}{\partial x}\right]$$

$$\frac{\partial}{\partial t}\left[\frac{\hbar}{2i\mu c^2}\left(\psi^*\frac{\partial \psi}{\partial t} - \psi\frac{\partial \psi^*}{\partial t}\right)\right] = \frac{\partial}{\partial x} j(x,t)$$

$$\Rightarrow \rho(x,t) = \frac{\hbar}{2i\mu c^2}\left(\psi\frac{\partial \psi^*}{\partial t} - \psi^*\frac{\partial \psi}{\partial t}\right)$$

pozitif tanımlı olmadığı için K-G denklemi Schrödinger denklemine alternatif bir tek parçacık denk.i için uygun değildir.

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prb.3.5.)
$$\psi(x) = \left(\frac{\pi}{\alpha}\right)^{-1/4} e^{-\alpha x^2/2}$$

a)
$$\langle x^4 \rangle = \left(\frac{\pi}{\alpha}\right)^{-1/2}\int_{-\infty}^{+\infty} dx\, x^4\, e^{-\alpha x^2}$$

$$= \left(\frac{\pi}{\alpha}\right)^{-1/2} \alpha^{-1/2}\, \alpha^{-4/2} \int_{-\infty}^{+\infty} dz\, z^4\, e^{-z^2}$$

$$\Gamma(z) = 2\int_0^{\infty} e^{-x^2} x^{2z-1}\, dx$$

$$\langle x^n \rangle = \frac{\alpha^{-n/2}}{\sqrt{\pi}} \left[\int_0^\infty dz\, z^n e^{-z^2} + (-1)^n \int_0^\infty dz\, z^n e^{-z^2} \right]$$

$$= \frac{\alpha^{-n/2}}{\sqrt{\pi}} \left[1 + (-1)^n \right] \Gamma[(n+1)/2]$$

$$= \begin{cases} 0 & n = 2m+1 \\ \frac{1}{\sqrt{\pi}} \alpha^{-m} \Gamma\left(m + \frac{1}{2}\right) = \frac{(2m-1)!!}{(2\alpha)^m} & n = 2m \end{cases}$$

b) $\langle x^2 \rangle = \frac{1}{2\alpha}$, $\langle x \rangle = 0$

$$\Delta x \equiv \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{1}{\sqrt{2\alpha}}$$

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prb. 3.6.) $\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dx\, e^{-ipx/\hbar} \left(\frac{\pi}{\alpha}\right)^{-1/4} e^{-\alpha x^2/2}$

$$= \left(\frac{\alpha}{\pi}\right)^{1/4} \frac{1}{\sqrt{2\pi\hbar}} e^{-p^2/2\alpha\hbar^2} \underbrace{\int_{-\infty}^{+\infty} dx\, e^{-\frac{\alpha}{2}\left[x + \frac{i\hbar}{\alpha}\right]^2}}_{\sqrt{2\pi/\alpha}}$$

$$= \left(\frac{1}{\pi\alpha\hbar^2}\right)^{1/4} e^{-p^2/2\alpha\hbar^2}$$

a) $\langle p^n \rangle = \left(\frac{1}{\pi\hbar^2\alpha}\right)^{1/2} \int dp\, p^n e^{-p^2/\alpha\hbar^2} = \begin{cases} 0 & n = 2m+1 \\ \frac{(\alpha\hbar^2)^m}{2^m}(2m-1)!! & n = 2m \end{cases}$

b) $\langle p^2 \rangle = \frac{\alpha\hbar^2}{2}$ $\langle p \rangle = 0$

$$\Delta p = \sqrt{\frac{\alpha}{2}}\,\hbar \qquad \Delta x \Delta p = \hbar\sqrt{\frac{\alpha}{2}} \cdot \frac{1}{\sqrt{2\alpha}} = \frac{\hbar}{2}$$

BS K

prb. 3.7.) $\psi(x) = \dfrac{N}{x^2+a^2}$

a)

$$N^2 \int_{-\infty}^{+\infty} dx \frac{1}{(x^2+a^2)^2} = N^2 \left[\frac{1}{2a^2} \frac{x}{x^2+a^2} + \frac{1}{2a^2} \frac{1}{a} \tan^{-1} \frac{x}{a} \right]_{-\infty}^{+\infty}$$

$$= N^2 \frac{\pi}{2a^3} \Rightarrow N = \sqrt{\frac{2a^3}{\pi}}$$

b) $\langle x^n \rangle = N^2 \displaystyle\int_{-\infty}^{+\infty} dx \frac{x^n}{x^2+a^2}$

$\lim\limits_{x\to\infty} \dfrac{x^n}{x^4}$ en az $\dfrac{1}{x^2}$ gibi davranmalı dolayısıyla

$n=1$ ve $n=2$ için yakınsak.

$n=1 \Rightarrow \langle x \rangle = 0$

$$n=2 \Rightarrow \langle x^2 \rangle = \frac{2a^3}{\pi} \int_{-\infty}^{+\infty} \frac{dx\, x^2}{(x^2+a^2)^2} = \frac{2a^3}{\pi} \left[\underbrace{\int \frac{dx}{x^2+a^2}}_{\pi/a} - a^2 \int \frac{dx}{(x^2+a^2)^2} \right]$$

$$= \frac{2a^3}{\pi} \left[\frac{\pi}{a} - a^2 \cdot \frac{\pi}{2a^3} \right] = a^2$$

c) $\langle p^2 \rangle = N^2 \displaystyle\int dx \frac{1}{x^2+a^2} \left(-\hbar^2 \frac{d^2}{dx^2} \right) \frac{1}{x^2+a^2}$

$$= -N^2 \hbar^2 2 \int dx \frac{3x^2-a^2}{(x^2+a^2)^4} = \frac{\hbar^2}{2a^2}$$

BS K

prb.3.8.) $e^{ipa/\hbar}\, x\, e^{-ipa/\hbar} = x + a$

$$e^{ipa/\hbar} x_{op} e^{-ipa/\hbar} f(p) = e^{ipa/\hbar}\left(i\hbar \frac{\partial}{\partial p}\right)\left(e^{-ipa/\hbar} f(p)\right)$$

$$= e^{ipa/\hbar}\, \cancel{i\hbar} \left(-\frac{ia}{\cancel{\hbar}}\right) e^{-ipa/\hbar} f(p) + e^{ipa/\hbar}\, i\hbar\, e^{-ipa/\hbar} \frac{df}{\partial p}$$

$$= a f(p) + i\hbar \frac{d}{dp} f$$

$$= (a + x_{op}) f(p)$$

---

prb.3.9.) $-\pi \le \theta \le +\pi$: $\psi(\theta)$ / $\psi(\pi) = \psi(-\pi)$

$$L = \frac{\hbar}{i}\frac{d}{d\theta}$$

$$\langle L\rangle - \langle L\rangle^* = \int_{-\pi}^{+\pi} d\theta\, \psi^*(\theta) \frac{\hbar}{i}\frac{d}{d\theta}\psi(\theta) - \int_{-\pi}^{+\pi} d\theta\, \psi(\theta)\left(-\frac{\hbar}{i}\frac{d}{d\theta}\right)\psi^*(\theta)$$

$$= \frac{\hbar}{i}\int_{-\pi}^{+\pi} d\theta \left(\psi^* \frac{d\psi}{d\theta} + \psi \frac{d\psi^*}{d\theta}\right) = \frac{\hbar}{i}\int_{-\pi}^{+\pi} d\theta \frac{d}{d\theta}|\psi|^2$$

$$= \frac{\hbar}{i}\left[|\psi|^2\right]_{-\pi}^{+\pi} = \frac{\hbar}{i}\left(\underbrace{|\psi(\pi)|^2 - |\psi(-\pi)|^2}_{0}\right)$$

---

prb.3.10) $\langle x\rangle - \langle x\rangle^* = i\hbar \int_0^\infty dp \left(\phi^* \frac{d\phi}{dp} + \phi \frac{d\phi^*}{dp}\right)$

$$= i\hbar \int_0^\infty dp \frac{d}{dp}|\phi(p)|^2$$

$$= i\hbar\left(|\phi(\infty)|^2 - |\phi(0)|^2\right)$$

$$= 0 \iff |\phi(\infty)|^2 = |\phi(0)|^2$$

BS K

prb. 3.2.) (3.11.) $\frac{\partial}{\partial t} P(x,t) + \frac{\partial}{\partial x} j(x,t) = 0 \Leftrightarrow V = V^*$

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi$$

ispat: $\frac{\partial}{\partial t}(\psi^* \psi) + \frac{\hbar}{2im} \frac{\partial}{\partial x}\left[\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right]$

$$= \psi \frac{\partial \psi^*}{\partial t} + \psi^* \frac{\partial \psi}{\partial t} + \frac{\hbar}{2im}\left[\cancel{\frac{\partial \psi^*}{\partial x}\frac{\partial \psi}{\partial x}} + \psi^* \frac{\partial^2 \psi}{\partial x^2} - \cancel{\frac{\partial \psi}{\partial x}\frac{\partial \psi^*}{\partial x}} - \psi \frac{\partial^2 \psi^*}{\partial x^2}\right]$$

$$= \left[\cancel{\frac{\hbar}{2im}\frac{\partial^2 \psi^*}{\partial x^2}} + i\frac{V^*}{\hbar}\psi^*\right]\psi + \psi^*\left[\cancel{-\frac{\hbar}{2im}\frac{\partial^2 \psi}{\partial x^2}} - i\frac{V}{\hbar}\psi\right]$$

$$+ \frac{\hbar}{2im}\left[\cancel{\psi^* \frac{\partial^2 \psi}{\partial x^2}} - \cancel{\psi \frac{\partial^2 \psi^*}{\partial x^2}}\right]$$

$$= \frac{i}{\hbar}(V^* - V)\psi^* \psi = 0 \Leftrightarrow V^* = V$$

prb. 3.3.) $V(x)$ kompleks olsun.

$$\Rightarrow \frac{\partial}{\partial t} P = \frac{\hbar}{2im}\left[\frac{\partial^2 \psi^*}{\partial x^2}\psi - \psi^* \frac{\partial^2 \psi}{\partial x^2}\right] + \frac{i}{\hbar}(V^* - V)\psi^* \psi$$

$$\int dx \frac{\partial P}{\partial t} = \frac{d}{dt}\int dx P = \overbrace{\frac{\hbar}{2im}\int dx \left[\psi \frac{\partial^2 \psi^*}{\partial x^2} - \psi^* \frac{\partial^2 \psi}{\partial x^2}\right]}^{=0}$$

$$+ \frac{i}{\hbar}\int dx (V^* - V)\psi^* \psi$$

$$\frac{d}{dt}\underbrace{\int dx P}_{} = \frac{2}{\hbar}\int dx |\psi|^2 \, \mathrm{Im} V$$

zamanla azalıyor ise

(yani soğuruluyor ise)

$\widetilde{\text{zaten}}$ negatif olmalı $\exists$ $\mathrm{Im} V < 0$

2

Pr. 3.4.) $\frac{1}{c^2}\frac{\partial^2\psi(x,t)}{\partial t^2} - \frac{\partial^2\psi(x,t)}{\partial x^2} + \left(\frac{\mu c}{\hbar}\right)^2\psi(x,t) = 0 \quad / \psi^*$

$$\frac{1}{c^2}\frac{\partial^2\psi^*}{\partial t^2} - \frac{\partial^2\psi^*}{\partial x^2} + \left(\frac{\mu c}{\hbar}\right)^2\psi^* = 0 \quad / \psi$$

$$\frac{1}{c^2}\left[\psi^*\partial_t^2\psi - \psi\partial_t^2\psi^*\right] = \psi^*\partial_x^2\psi - \psi\partial_x^2\psi^*$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial}{\partial t}\left[\psi^*\partial_t\psi - \psi\partial_t\psi^*\right] = \partial_x\left[\psi^*\partial_x\psi - \psi\partial_x\psi^*\right]$$

$\hbar/2i\mu$

$$\frac{\partial}{\partial x}\left[\frac{\hbar}{2i\mu}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right)\right] = \frac{\partial}{\partial t}\left[\frac{\hbar}{2i\mu c^2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right)\right]$$

$$P(x,t) = \frac{\hbar}{2i\mu c^2}\left(\psi\frac{\partial\psi^*}{\partial t} - \psi^*\frac{\partial\psi}{\partial t}\right)$$

pozitif tanımlı değil! K-G. denk. i tek parçacık denklemi olarak uygun değil.

BS K

$2z-1$ $\quad 3$

$$\Gamma(z) = 2\int_0^\infty e^{-x^2} x^{2z-1}\,dx$$

Prb. 3.5.) $\quad \psi(x) = \left(\frac{\pi}{\alpha}\right)^{-1/4} e^{-\alpha x^2/2}$

a) $$\langle x^n \rangle = \left(\frac{\pi}{\alpha}\right)^{-1/2} \int dx\, x^n\, e^{-\alpha x^2}$$

$$= \left(\frac{\pi}{\alpha}\right)^{-1/2} \alpha^{-1/2} \alpha^{-n/2} \int_{-\infty}^{+\infty} dz\, z^n e^{-z^2}$$

$$= \frac{1}{\sqrt{\pi}} \alpha^{-n/2} \left[1+(-1)^n\right] \Gamma\left(\frac{n+1}{2}\right)$$

$$= \begin{cases} 0 & n = 2m+1 \\ \frac{1}{\sqrt{\pi}} \alpha^{-m} \Gamma\left(m+\frac{1}{2}\right) = \frac{(2m-1)!!}{(2\alpha)^m} & n = 2m \end{cases}$$

b) $\langle x^2 \rangle = \frac{1}{2\alpha}$, $\quad \langle x \rangle = 0$

$$(\Delta x)^2 = \left[\langle x^2\rangle - \langle x \rangle^2\right] = \frac{1}{\sqrt{2\alpha}}$$

prb. 3.6.)

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dx\, e^{-ipx/\hbar} \left(\frac{\pi}{\alpha}\right)^{-1/4} e^{-\alpha x^2/2}$$

$$= \left(\frac{\alpha}{\pi}\right)^{1/4} \frac{1}{(2\pi\hbar)^{1/2}} e^{-p^2/2\alpha\hbar^2} \underbrace{\int dx\, e^{-\frac{\alpha}{2}\left(x+\frac{ik}{\alpha}\right)^2}}_{(2)\sqrt{\frac{2\pi}{\alpha}}}$$

$$= \left(\frac{1}{\pi\alpha\hbar^2}\right)^{1/4} e^{-p^2/2\alpha\hbar^2}$$

$$\langle p^n \rangle = \left(\frac{1}{\pi\alpha\hbar^2}\right)^{1/2} \int dp\, p^n e^{-p^2/\alpha\hbar^2} = \begin{cases} 0 & n = 2m+1 \\ \frac{(\alpha\hbar^2)^m (2m-1)!!}{2^m} & n = 2m \end{cases}$$

b) $\langle p^2 \rangle = \frac{\alpha\hbar^2}{2}$ $\quad \langle p \rangle = 0$ $\quad \Delta p = \sqrt{\frac{\alpha}{2}}\,\hbar$

BS K

$$\Delta x \Delta p = \hbar \sqrt{\frac{\alpha}{2}}\,\hbar\, \frac{1}{\sqrt{2\alpha}} = \hbar/2$$

prb. 3.7) $\psi = \frac{N}{x^2+a^2}$

$$N^2 \int_{-\infty}^{+\infty} dx \frac{1}{(x^2+a^2)^2} = N^2 \left[\frac{1}{2a^2} \frac{x}{x^2+a^2} + \frac{1}{2a^2} \frac{1}{a} \arctan \frac{x}{a} \right]_{-\infty}^{+\infty}$$

$$= N^2 \frac{\pi}{2a^3} = 1$$

$$N = \sqrt{\frac{2a^3}{\pi}}$$

b) $\langle x^n \rangle = N^2 \int dx \frac{x^n}{(x^2+a^2)^2}$ $\quad n=1$ ve 2 için yakınsar.

x^4/x^4 için $1/x^2$ gibi davranır

$n=1 \Rightarrow \langle x \rangle = 0$

$n=2 \Rightarrow \langle x^2 \rangle = \frac{2a^3}{\pi} \int \frac{dx\, x^2}{(x^2+a^2)^2} = \frac{2a^3}{\pi} \left[\underbrace{\int \frac{dx}{x^2+a^2}}_{\pi/a} - a^2 \int \frac{dx}{(x^2+a^2)^2} \right]$

$$= \frac{2a^3}{\pi} \left[\frac{\pi}{a} - a^2 \frac{\pi}{2a^3} \right] = a^2$$

c) $\langle p^2 \rangle = \int dx \frac{1}{x^2+a^2} \left(-\hbar^2 \frac{d^2}{dx^2} \frac{1}{x^2+a^2} \right) = -N^2 \hbar^2 2 \int dx \frac{3x^2 - a^2}{(x^2+a^2)^4}$

$$= \hbar^2 / 2a^2$$

Ya da

2.2 problem

$$\phi(p) = \frac{N}{\sqrt{2\pi\hbar}} \int dx \frac{e^{-ipx/\hbar}}{x^2+a^2} = \frac{N}{\sqrt{2\pi\hbar}} \frac{\pi}{a} e^{-a|p|/\hbar}$$

$$\langle p^2 \rangle = \frac{N^2 \pi^2}{2\pi\hbar a^2} \int_{-\infty}^{+\infty} dp\, p^2 e^{-2a|p|/\hbar} = \frac{N^2 \pi}{\hbar a^2} \int_0^{\infty} dp\, e^{-2ap/\hbar} p^2 = \frac{\hbar^2}{2a^2}$$

$\Delta x = a \quad \Delta p = \frac{\hbar}{\sqrt{2}a} \quad \Delta x \, \Delta p = \hbar/\sqrt{2}$

BS K

prb. 3.8.)

$$e^{ipa/\hbar} X_{op}\, e^{-ipa/\hbar} = x + a$$

$$e^{ipa/\hbar}\, i\hbar \frac{d}{dp}\, e^{-ipa/\hbar} f(p) = e^{ipa/\hbar}\, i\hbar \left(-\frac{ia}{\hbar}\right) e^{-ipa/\hbar} f(p)$$
$$+\, e^{ipa/\hbar}\, i\hbar\, e^{-ipa/\hbar} \frac{df}{dp}$$

$$\Rightarrow \quad = a f + i\hbar \frac{d}{dp} f = (a + x_{op}) f$$

prb. 3.9.) $\quad \psi(\pi) = \psi(-\pi) \qquad L = \frac{\hbar}{i}\frac{d}{d\theta}$

$$\langle L\rangle - \langle L\rangle^* = \int_{-\pi}^{+\pi} d\theta\, \psi^*(\theta)\, \frac{\hbar}{i}\frac{d}{d\theta}\psi(\theta) - \int_{-\pi}^{+\pi} d\theta\, \psi(\theta)\left(-\frac{\hbar}{i}\frac{d}{d\theta}\right)\psi^*(\theta)$$

$$= \frac{\hbar}{i}\int_{-\pi}^{+\pi} d\theta \left(\psi^* \frac{d\psi}{d\theta} + \psi \frac{d\psi^*}{d\theta}\right) = \frac{\hbar}{i}\int_{-\pi}^{+\pi} d\theta \frac{d}{d\theta}|\psi|^2$$

$$\Rightarrow \quad = \frac{\hbar}{i}\left[|\psi(\pi)|^2 - |\psi(-\pi)|^2\right] = 0$$

BS K