BÖLÜM 7

KUANTUM MEKANİĞİNDE İŞLEMCİ YÖNTEMLERİ

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 \qquad [p, x] = \frac{\hbar}{i}$$

$$H = \omega \left(\sqrt{\frac{m\omega}{2}}\, x - i \frac{p}{\sqrt{2m\omega}} \right) \left(\sqrt{\frac{m\omega}{2}}\, x + i \frac{p}{\sqrt{2m\omega}} \right)$$

klasik olarak, fakat p ve x yerdeğiştirmediğinden,

$$\omega \left(\sqrt{\frac{m\omega}{2}}\, x - i \frac{p}{\sqrt{2m\omega}} \right) \left(\sqrt{\frac{m\omega}{2}}\, x + i \frac{p}{\sqrt{2m\omega}} \right)$$

$$= \omega \left[\frac{1}{2} m\omega x^2 + \frac{p^2}{2m\omega} - i \frac{1}{2} (px - xp) \right]$$

$$= \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 - \frac{i\omega}{2} (px - xp)$$

$$= H - \frac{i\omega}{2} [p, x] = H - \frac{\hbar\omega}{2}$$

$$A = \sqrt{\frac{m\omega}{2}}\, x + i \frac{p}{\sqrt{2m\omega}} \qquad A^{+} = \sqrt{\frac{m\omega}{2}}\, x - i \frac{p}{\sqrt{2m\omega}}$$

$$[A, A^{+}] = \left[\sqrt{\frac{m\omega}{2}}\, x + i \frac{p}{\sqrt{2m\omega}} \,,\, \sqrt{\frac{m\omega}{2}}\, x - i \frac{p}{\sqrt{2m\omega}} \right]$$

$$= -\frac{1}{2} i \underbrace{[x, p]}_{-\frac{\hbar}{i}} + \frac{1}{2} i \underbrace{[p, x]}_{\hbar/i} = \hbar$$

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$$H = \omega A^+ A + \frac{\hbar\omega}{2}$$

$$= \left(\frac{A^+A}{\hbar} + \frac{1}{2}\right)\hbar\omega$$

$$[H, A] = [\omega A^+A, A] = \omega\left(A^+\cancel{[A,A]} + \underbrace{[A^+,A]}_{-\hbar} A\right)$$

$$= -\omega\hbar A$$

$$[H, A^+] = [\omega A^+A, A^+] = \omega A^+[A, A^+]$$

$$= \hbar\omega A^+$$

$$[A,B]^+ = (AB - BA)^+ = B^+A^+ - A^+B = [B^+, A^+]$$

$$[H,A]^+ = [A^+, H] = -[H, A^+] = (-\hbar\omega A)^+$$

$HU_E = EU_E$ özdeğer denk.

$\int$ karesi integre edilebilir.

Özfonk. lar $\longrightarrow$ özdurumlar

dalga fonk. ları $\longrightarrow$ durum vektörleri.

$[H,A]U_E = -\hbar\omega AU_E$ idi

$HAU_E - AHU_E = -\hbar\omega AU_E$

$$\mathcal{H} A U_E = (E - \hbar\omega) A U_E$$

U_E : $\mathcal{H}$ 'nin E özdeğerli özdurumu

AU_E de " $(E-\hbar\omega)$ " " olur.

* $\mathcal{E} = \hbar\omega$. enerji bu kadar azalmıştır. Bu yüzden,

$$A U_E = C(E)\, U_{E-\mathcal{E}}$$

(U_E. 1'e normalize olmasına karşın $U_{E-\varepsilon}$'nin (AU_E'nin) 1'e normalize olması gerekmez.)

$$\int U_E^*(x) U_E(x)\, dx = 1$$

$$\longrightarrow \langle U_E | U_E \rangle = 1$$

$$\langle U_E | U_{E'} \rangle = \begin{cases} \delta(E-E') \\ \delta(p-p') \end{cases}$$ Sürekli.

$$[\mathcal{H}, A]\, U_{E-\mathcal{E}} = -\hbar\omega A U_{E-\mathcal{E}}$$

$$\mathcal{H} A U_{E-\mathcal{E}} - A\mathcal{H} U_{E-\mathcal{E}} = -\hbar\omega A U_{E-\mathcal{E}}$$

$$\mathcal{H}(A U_{E-\mathcal{E}}) = (E - 2\mathcal{E}) U_{E-\mathcal{E}}$$ $\therefore$ A azaltma işlemcisi.

BS K

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A'nın kaç kez uygulanabileceğinin bir sınırı vardır.

Keyfi bir ψ için;

$$\langle\psi|p^2|\psi\rangle = \int \psi^*(x)\, p^2 \psi(x)\, dx$$

$$= \int dx\, [p^\dagger \psi(x)]\, [p\psi(x)]$$

$$= \int dx\, [p\psi(x)]^*\, [p\psi(x)]$$

$$= \int dx \left| \frac{d\psi(x)}{dx} \right|^2 \geq 0$$

$$\langle\psi|p^2|\psi\rangle = \langle p^\dagger\psi|p\psi\rangle$$

$$= \langle p\psi|p\psi\rangle \geq 0$$

<u>p ve x hermitsel</u>

$$\langle\psi|x^2|\psi\rangle = \langle x^\dagger\psi|x\psi\rangle$$

$$= \langle x\psi|x\psi\rangle \geq 0$$

Dolayısıyla $\langle\psi|H|\psi\rangle \geq 0$ olmalı. Yani A'nın kaç kere uygulanabileceğinin sınırı buradan çıkar.

Taban durumunda işlem son bulmalı.

BS K

$$A u_0 = 0 \quad \text{olmalı.}$$

$$H U_0 = \left(\omega A^{\dagger} A + \frac{1}{2}\hbar\omega\right) U_0 = \underbrace{\frac{\hbar\omega}{2}}_{\text{T.D. enerjisi}} U_0 \quad \text{olur.}$$

$$[H, A^{\dagger}] = \hbar\omega A^{\dagger} \quad \text{idi.}$$

$$[H, A^{\dagger}] U_0 = \hbar\omega A^{\dagger} U_0 \quad \Big(= H A^{\dagger} U_0 - A^{\dagger} \underbrace{H U_0}_{\frac{\hbar\omega}{2} U_0} \Big)$$

$$H A^{\dagger} U_0 = \left(\hbar\omega + \frac{\hbar\omega}{2}\right) A^{\dagger} U_0$$

$\therefore$ enerji $\hbar\omega = \varepsilon$ birimi kadar artmıştır. $A^{\dagger}$ arttırma işlemcisi.

$$\begin{cases} A^{\dagger} U_0 = C\, U_1 \\ A\, U_1 = C' U_0 \end{cases}$$

$\therefore$ $A^{\dagger}$'ı U_0'a ardarda uygulayarak tüm durumları bulabiliriz.

$$E = \left(n + \frac{1}{2}\right)\hbar\omega \qquad n = 0, 1, 2, \ldots$$

$$u_n = \frac{1}{\sqrt{n!}} \left(\frac{A^{\dagger}}{\sqrt{\hbar}}\right)^n U_0$$

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$$A^{\dagger n} u_0 = C_n u_n$$

$$\Rightarrow \quad |C_n|^2 \langle u_n | u_n \rangle = \langle A^{\dagger n} u_0 | A^{\dagger n} u_0 \rangle$$

$$= \langle u_0 | A^n A^{\dagger n} u_0 \rangle$$

$$A^n A^{\dagger n} = A^{n-1} (\underbrace{A A^\dagger}) A^{\dagger n-1} \qquad [A, A^\dagger] = \hbar$$

$$= A^{n-1} (A^\dagger A + \hbar) A^{\dagger n-1}$$

$$= A^{n-1} A^\dagger (\underbrace{A A^\dagger}_{A^\dagger A + \hbar} + \hbar) A^{\dagger n-2}$$

$$= A^{n-1} A^\dagger (A^\dagger A + 2\hbar) A^{\dagger n-2}$$

$$= A^{n-1} A^{\dagger n-1} (A^\dagger A + n\hbar)$$

$$= A^{n-1} (n\hbar A^{\dagger n-1} + A^{\dagger n} A)$$

$$\langle u_0 | A^n A^\dagger u_0 \rangle = \langle u_0 | A^{n-1} n\hbar A^{\dagger n-1} u_0 \rangle$$

$$= n\hbar \langle A^{\dagger n-1} u_0 | A^{\dagger n-1} u_0 \rangle$$

$$= n\hbar |C_{n-1}|^2$$

$$= n(n-1)\hbar^2 |C_{n-2}|^2$$

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$$= n! \hbar^n |C_0|^2 = n! \hbar^n$$

$$\langle u_0 | A^+ u_0 \rangle = (A u_0 | u_0 \rangle = 0$$

$$= \langle u_1 | u_0 \rangle = 0$$

$$\langle u_m | u_n \rangle = 0 \quad m \neq n.$$

Herhangi bir ψ için

$$\psi = \sum_{n=0}^{\infty} C_n u_n$$

$$C_m = \langle u_m | \psi \rangle$$

$A u_0 = 0$ ise

$$\left(\sqrt{\frac{m\omega}{2}}\, x + i \frac{p}{\sqrt{2m\omega}} \right) u_0 = 0$$

$$\left(m\omega x + \hbar \frac{d}{dx} \right) u_0(x) = 0$$

$$u_0(x) = C e^{-m\omega x^2 / 2\hbar}$$

$$\int_{-\infty}^{+\infty} u_0^*(x)\, u_0(x)\, dx = 1$$

$$|C|^2 \int_{-\infty}^{+\infty} dx\, e^{-m\omega x^2/\hbar} = 1 = |C|^2 \left(\frac{\hbar\pi}{m\omega} \right)^{1/2}$$

BS K

$$C = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4}$$

$$\psi_n(x) = \frac{1}{\sqrt{\hbar^n n!}} A^{\dagger^n} \psi_0$$

$$= \frac{1}{\sqrt{n!}} \hbar^{-n/2} \left(\sqrt{\frac{m\omega}{2}}\, x - i\sqrt{\frac{1}{2m\omega}}\, p\right)^n e^{-m\omega x^2/2\hbar}$$

$$= \frac{1}{\sqrt{n!}} \left(\sqrt{\frac{m\omega}{2\hbar}}\, x - \sqrt{\frac{\hbar}{2m\omega}} \frac{d}{dx}\right)^n e^{-m\omega x^2/2\hbar}$$

$\star$

$$X_{op} \phi_x = x \phi_x$$

$$\psi = \int dx\, C(x)\, \phi_x$$

$$\langle \phi_x | \phi_{x'} \rangle = 0 \quad \text{orthogonal}$$

$$C_x = \langle \phi_x | \psi \rangle$$

BS K

$$i\hbar \frac{d\psi(t)}{dt} = H\psi(t)$$

$$\psi(t) = e^{-iHt/\hbar}\,\psi(0)$$

$$= \sum_{n=0}^{\infty} \frac{(-iHt)^n}{\hbar^n\, n!}\,\psi(0)$$

$$\langle A \rangle_t = \langle \psi(t) \,|\, A\psi(t) \rangle$$

$$= \langle e^{-iHt/\hbar}\psi(0) \,|\, A\, e^{-iHt/\hbar}\psi(0) \rangle$$

$$= \langle \psi(0) \,|\, A(t)\,\psi(0) \rangle$$

$$A(t) = e^{iHt/\hbar} A\, e^{-iHt/\hbar}$$ Heisenberg temsili

↳ zamandan bağımsız Schrödinger gösterimi

$$\frac{d}{dt}A(t) = \frac{iH}{\hbar} e^{iHt/\hbar} A\, e^{-iHt/\hbar} - \frac{i}{\hbar} e^{iHt/\hbar} AH\, e^{-iHt/\hbar}$$

$$= \frac{i}{\hbar} H A(t) - \frac{i}{\hbar} A(t) H$$

$$= \frac{i}{\hbar}\left[H, A(t)\right]$$

$$H = \omega A^{\dagger} A + \frac{\hbar\omega}{2}$$

$e^{+iHt/\hbar}$ $\qquad$ $e^{-iHt/\hbar}$

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$$\mathcal{H} = \omega A^{\dagger}(t) A(t) + \frac{1}{2}\hbar\omega$$

$$[A(t), A^{\dagger}(t)] = \hbar$$

$$\left.\begin{aligned} \frac{d}{dt} A(t) &= -i\omega A(t) \\ \frac{d}{dt} A^{\dagger}(t) &= i\omega A^{\dagger}(t) \end{aligned}\right\} \quad \begin{aligned} A(t) &= e^{-i\omega t} A(0) \\ A^{\dagger}(t) &= e^{i\omega t} A^{\dagger}(0) \end{aligned}$$

$$p(t) = p(0) \cos\omega t - m\omega x(0) \sin\omega t$$

$$x(t) = x(0) \cos\omega t + \frac{p(0)}{m\omega} \sin\omega t$$

BS K

BÖLÜM 7

PROBLEMLER

prb 7.1.) $[A, A^+] = \hbar$ (7.5), $U_n = \frac{1}{\sqrt{n!}}\left(\frac{A^+}{\sqrt{\hbar}}\right)^n U_0$ 7.26.

$$A U_n = \sqrt{n\hbar}\, U_{n-1}$$

$$A U_{n+1} = A \frac{1}{\sqrt{(n+1)!}}\left(\frac{A^+}{\sqrt{\hbar}}\right)^{n+1} U_0 = \frac{AA^+}{\sqrt{(n+1)\hbar}} U_n$$

$$= \frac{1}{\sqrt{(n+1)\hbar}}(A^+A + \hbar)U_n = \frac{1}{\sqrt{(n+1)\hbar}}\left\{A^+\sqrt{n\hbar}\, U_{n-1} + \hbar U_n\right\}$$

$$= \frac{1}{\sqrt{(n+1)\hbar}}\left\{n\hbar U_n + \hbar U_n\right\} = \sqrt{(n+1)\hbar}\, U_n$$

$n=1$ için $A U_1 = A\frac{A^+}{\sqrt{\hbar}} U_0 = \frac{1}{\sqrt{\hbar}} AA^+ U_0 = \frac{1}{\sqrt{\hbar}}(A^+A + \hbar)U_0 = \sqrt{\hbar}\, U_0$

(A) $[A, A^{+n}] = n\hbar A^{+\,n-1}$ 7.5 den

$$[A, A^{+n}]U_0 = n\hbar A^{+\,n-1} U_0 \rightarrow AA^{+n}U_0 = n\hbar A^{+\,n-1} U_0$$

$$A\sqrt{n!}\,\hbar^{n/2} U_n = n\hbar\sqrt{(n-1)!}\,\hbar^{(n-1)/2} U_{n-1}$$

$$\Rightarrow A U_n = \sqrt{n\hbar}\, U_{n-1} \checkmark$$

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prb. 7.2.) $f(A^+) = \sum_{n=1}^{N} a_n A^{+n}$ olsun.

$$A f(A^+) U_0 = \sum a_n A A^{+n} U_0$$

$$= \sum a_n [\cancel{A^{+n} A} + n\hbar A^{+n-1}] U_0$$

$$= \hbar \sum a_n n A^{+n-1} U_0 = \hbar \frac{d}{dA^+} \sum_{n=1}^{N} a_n A^{n}$$

$$= \hbar \frac{d f(A^+)}{dA^+} U_0$$

$$A = \hbar \frac{d}{dA^+} \qquad [A, A^+] = \hbar \; , \; [p, x] = \frac{\hbar}{i}$$

prb. 7.3.) $x = \frac{1}{\sqrt{2M\omega}} (A + A^+)$

$$\langle U_n | x | U_m \rangle = \frac{1}{\sqrt{2M\omega}} \langle U_n | A | U_m \rangle + \frac{1}{\sqrt{2M\omega}} \langle U_n | A^+ | U_m \rangle$$

$$= \frac{\sqrt{m\hbar}}{\sqrt{2M\omega}} \delta_{n,m-1} + \frac{\sqrt{n\hbar}}{\sqrt{2M\omega}} \delta_{n-1,m}$$

$$= \begin{cases} 0 & \text{diğer haller} \\ \neq 0 & n = m \pm 1 \end{cases}$$

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prb. 7.4.) $e^{\lambda A} f(A^+) U_0 = \sum \frac{\lambda^n}{n!} A^n f(A^+) U_0$

$= \sum \frac{\lambda^n}{n!} A^{n-1} A f(A^+) U_0$

$= \sum \frac{\lambda^n}{n!} A^{n-1} \hbar \frac{d f(A^+)}{d A^+} U_0 = \sum \frac{\lambda^n}{n!} \hbar^n \frac{d^n f(A^+)}{d A^{+n}} U_0$

$= f(A^+ + \lambda \hbar) U_0$

prb. 7.5.) $e^{\lambda A} f(A^+) e^{-\lambda A} = f(A^+ + \lambda \hbar)$

$e^{\lambda A} f(A^+) \underbrace{e^{-\lambda A} g(A^+) U_0} = e^{\lambda A} \underbrace{f(A^+) g(A^+ - \lambda \hbar) U_0}_{F(A^+) U_0}$

$= F(A^+ + \lambda \hbar) U_0$

$= f(A^+ + \lambda \hbar) g(A^+ \cancel{- \lambda \hbar} \cancel{+ \lambda \hbar}) U_0 = f(A^+ + \lambda \hbar) g(A^+) U_0$

$\Rightarrow e^{\lambda A} f(A^+) e^{-\lambda A} = f(A^+ + \lambda \hbar)$

prb. 7.6.) $e^{aA + bA^+} = e^{aA} e^{bA^+} e^{-\frac{1}{2} ab \hbar}$

$e^{\lambda (aA + bA^+)} \equiv e^{\lambda a A} F(\lambda)$ deyip $F(\lambda)$'yı bulalım.

λ'ya göre türev alalım!

$(aA + bA^+) e^{\lambda (aA + bA^+)} = aA e^{\lambda (aA + bA^+)} + bA^+ e^{\lambda (aA + bA^+)}$

$= aA e^{\lambda a A} F(\lambda) + e^{\lambda a A} \frac{dF}{d\lambda}$

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$$(aA+bA^{\dagger})\,e^{\lambda aA}F(\lambda)=aA\,e^{\lambda aA}F(\lambda)+e^{\lambda aA}\frac{dF}{d\lambda}$$

$$e^{-\lambda aA}$$

$$e^{-\lambda aA}(\cancel{aA}+bA^{\dagger})\,e^{\lambda aA}F(\lambda)=\cancel{a\,e^{-\lambda aA}A\,e^{\lambda aA}F}+\frac{dF}{d\lambda}$$

$$\frac{dF}{d\lambda}=b\,\underbrace{e^{-\lambda aA}A^{\dagger}e^{\lambda aA}}_{(A^{\dagger}-\lambda a\hbar)}F(\lambda)$$

$$=b\,(A^{\dagger}-\lambda a\hbar)\,F(\lambda)$$

$$\ln F=bA^{\dagger}\lambda-ab\hbar\frac{\lambda^{2}}{2}\qquad F(\lambda)=e^{\lambda bA^{\dagger}}e^{-\frac{ab\hbar\lambda^{2}}{2}}$$

$$\lambda=1\Rightarrow\quad e^{aA+bA^{\dagger}}=e^{aA}\,e^{bA^{\dagger}}\,e^{-ab\hbar/2}\qquad \text{BCH.}$$

Prb. 7.7.

$$f(A^{\dagger})=[f(A)]^{\dagger}$$

$e^{\lambda A}f(A^{\dagger})e^{-\lambda A}=f(A^{\dagger}+\lambda\hbar)$ için $\dagger$'ını alalım.

$e^{-\lambda A^{\dagger}}f(A)\,e^{\lambda A^{\dagger}}=f(A+\lambda\hbar)$ bulunur.

$e^{\lambda(aA+bA^{\dagger})}=e^{\lambda bA^{\dagger}}F(\lambda)$ diyerek Prb. 7.6. daki

yol izlenirse

$$e^{aA+bA^{\dagger}}=e^{bA^{\dagger}}\,e^{aA}\,e^{\frac{1}{2}ab\hbar}\,.$$

prb. 7.8.

$$x = \frac{1}{\sqrt{2m\omega}}(A+A^\dagger)$$

$$e^{ikx} = e^{i\alpha(A+A^\dagger)} = e^{i\alpha A^\dagger} e^{i\alpha A} e^{-\alpha^2\hbar/2} \qquad \alpha = \frac{k}{\sqrt{2m\omega}}$$

$$\langle u_0|e^{ikx}|u_0\rangle = e^{-\alpha^2\hbar/2}\langle u_0|e^{i\alpha A^\dagger} e^{i\alpha A}|u_0\rangle$$

$$= e^{-\alpha^2\hbar/2}\langle u_0|u_0\rangle = e^{-k^2\hbar/4m\omega}$$

prb. 7.9.)

$$\int dx\, u_0^*(x)\, e^{ikx}\, u_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\int dx\, e^{-\frac{m\omega x^2}{\hbar}+ikx}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-k^2\hbar/4m\omega}\int_{-\infty}^{+\infty} dx\, e^{-\frac{m\omega}{\hbar}\left(x-\frac{ik\hbar}{2m\omega}\right)^2}$$

$$= e^{-k^2\hbar/4m\omega}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left(\frac{\pi\hbar}{m\omega}\right)^{1/2} = e^{-k^2\hbar/4m\omega}.$$

prb. 7.11.)

$$H = \frac{p^2(t)}{2m} + mg\,x(t) \qquad x(t) = ?$$

$$\frac{dx(t)}{dt} = \frac{i}{\hbar}[H, x(t)] = \frac{i}{\hbar}\frac{1}{2m}[p^2, x] = \frac{p(t)}{m}$$

$$\frac{dp(t)}{dt} = \frac{i}{\hbar}[H, p(t)] = \frac{i}{\hbar}\, mg\,[x, p] = -mg$$

$$\frac{d^2x}{dt^2} = -g \qquad \frac{dx}{dt} = -gt + v_0 \Rightarrow x(t) = -\frac{gt^2}{2} + v_0 t + x_0.$$

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prb. 7.11.) $H = \frac{p^2(t)}{2m} + \frac{1}{2} m\omega^2 x^2(t) - e\mathcal{E}\, x(t)$

$$\frac{dx}{dt} = \frac{i}{\hbar}[H, x] = \frac{p(t)}{m} \; ; \; \frac{dp}{dt} = \frac{i}{\hbar}[H, p] = -m\omega^2 x + e\mathcal{E}$$

$$\frac{d^2 p}{dt^2} = -m\omega^2 p(t) \Rightarrow p(t) = A e^{i\omega t} + B e^{-i\omega t}$$

$$\frac{dx}{dt} = \frac{A}{m} e^{i\omega t} + \frac{B}{m} e^{-i\omega t} \Rightarrow x(t) = A' e^{i\omega t} + B' e^{-i\omega t} + \frac{e\mathcal{E}}{m\omega^2}$$

$$x(0) = A' + B' + \frac{e\mathcal{E}}{m\omega^2} \qquad \frac{p(0)}{m} = i\omega(A' - B')$$

$$x(t) = (A' + B')\cos\omega t + i(A' - B')\sin\omega t + \frac{e\mathcal{E}}{m\omega^2}$$

$$x(t) = \left[x(0) - \frac{e\mathcal{E}}{m\omega^2}\right]\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t + \frac{e\mathcal{E}}{m\omega^2}$$

$$[x(t_1), x(t_2)] \neq 0.$$

BS K