BÖLÜM 10

RADYAL DENKLEM

$$\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)R_{n\ell}(r)-\frac{2\mu}{\hbar^2}\left[V(r)+\frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\right]R_{n\ell}(r)$$

$$+\frac{2\mu E}{\hbar^2}R_{n\ell}(r)=0$$

Denklemde m bağımlılığı yok, böylece verilen bir ℓ için $(2\ell+1)$ katlı dejenerelik var.

Coulomb pot. hariç, $1/r$ den daha hızlı sıfıra giden pot. ler için bu denklemi çözeceğiz. Ve orijinde,

$$\lim_{r\to 0} r^2 V(r)=0$$

varsayacağız.

$$u_{n\ell}(r)=r R_{n\ell}(r)$$

dönüşümünü yapalım. O zaman

$$\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)\frac{u_{n\ell}}{r}=\frac{1}{r}\frac{d^2u}{dr^2}.$$

$$\frac{d}{dr}\frac{u}{r}=-\frac{1}{r^2}u+\frac{1}{r}\frac{du}{dr}\Rightarrow\frac{d^2}{dr^2}\frac{u}{r}=\frac{2}{r^3}u-\frac{1}{r^2}\frac{du}{dr}-\frac{1}{r^2}\frac{du}{dr}+\frac{1}{r}\frac{d^2u}{dr^2}$$

$$\cancel{\frac{2}{r^3}u}\cancel{-\frac{2}{r^2}\frac{du}{dr}}+\frac{1}{r}\frac{d^2u}{dr^2}\cancel{+\frac{2}{r^2}\frac{du}{dr}}\cancel{-\frac{2}{r^3}u}$$

2

$$\cancel{\frac{1}{r}} \frac{d^2 u_{nl}}{dr^2} + \frac{2\mu}{\hbar^2}\left[E - V(r) - \frac{l(l+1)\hbar^2}{2\mu r^2}\right] \frac{u_{nl}(r)}{\cancel{r}} = 0$$

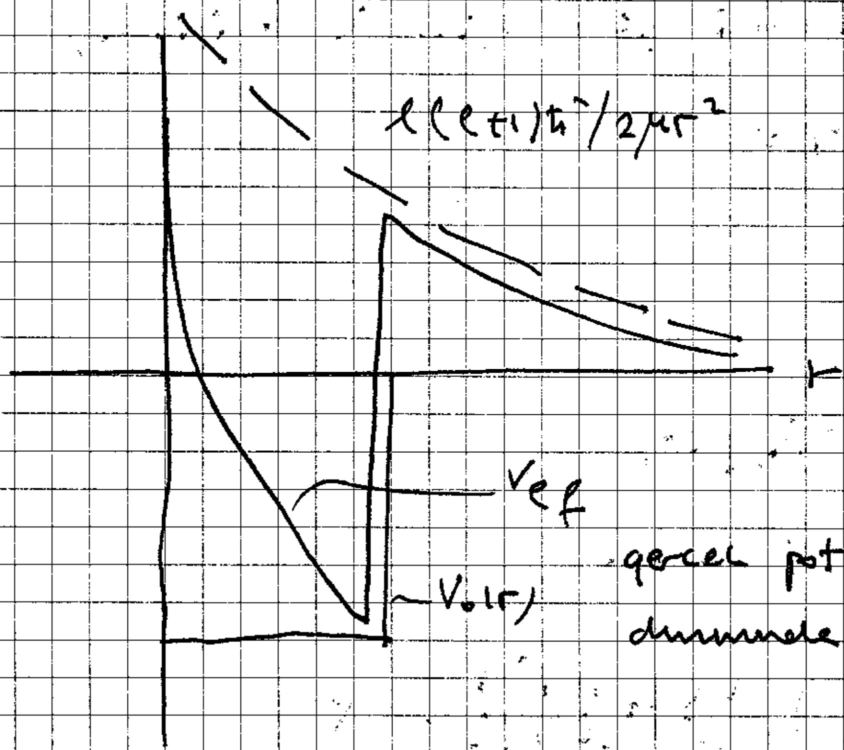
(a)

$$V(r) \longrightarrow V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2}$$

(b) Orijinde dalga fonk.'nun sonlu olması

$$u_{nl}(0) = 0$$

gerektirir.



gerçek pot.'ın kare kuyu olması durumunda

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$r \to 0$ civarında çözüme bakalım, burada etkin terim $1/r^2$ 'dir.

$$\frac{d^2u}{dr^2} - \frac{\ell(\ell+1)}{r^2}u \simeq 0$$

Çözüm önerisi $u(r) \sim r^s$

$$s(s-1) - \ell(\ell+1) = 0 \Rightarrow \begin{cases} s = \ell - 1 & u \sim r^{-\ell} \text{ düzensiz} \\ s = -\ell & u \sim r^{\ell-1} \text{ düzenli} \end{cases}$$

$(u(0) = 0$ için$)$

$r \to \infty$ iken

$$\frac{d^2u}{dr^2} + \frac{2\mu E}{\hbar^2}u \simeq 0$$

kareci integrallenebilirlik şartı

$$1 = \int d^3\vec{r}\,|\psi(\vec{r})|^2 = \int_0^\infty r^2 dr \int d\Omega\, |R_{n\ell} Y_{\ell m}|^2$$

$$\Rightarrow \int_0^\infty |r R_{n\ell}|^2 dr = 1$$

$$\int_0^\infty |u_{n\ell}|^2 dr = 1 \Rightarrow u \to 0 \text{ olmalı } r \to \infty \text{ iken}$$

$\frac{2\mu E}{\hbar^2} = -\alpha^2$ alırsak $E < 0$

$u(r) \sim e^{-\alpha r}$ ıraksamaz.

$\frac{2\mu E}{\hbar^2} = k^2$ alırsak $E > 0$

$u(r) \sim e^{\pm ikr}$

BS K

A — SERBEST PARÇACIK

$V(r) = 0$ ama hala $\dfrac{\ell(\ell+1)\hbar^2}{2\mu r^2}$ merkezkaç engeli mevcut.

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2}\right] R(r) + k^2 R(r) = 0$$

$$k^2 = 2\mu E/\hbar^2$$

$\rho = kr$ dönüşümü yapalım.

$$\frac{d}{dr} = \frac{d\rho}{dr}\frac{d}{d\rho} = k\,\frac{d}{d\rho}$$

$$\frac{d^2}{dr^2} = k\,\frac{d}{dr}\,\frac{d}{d\rho} = k^2\,\frac{d^2}{d\rho^2}$$

$$\left[\cancel{k^2}\,\frac{d^2}{d\rho^2} + \frac{2}{\rho}\,\cancel{k^2}\,\frac{d}{d\rho} - \frac{\ell(\ell+1)}{\rho^2}\,\cancel{k^2}\right] R + \cancel{k^2} R = 0$$

$$\left[\frac{d^2}{d\rho^2} + \frac{2}{\rho}\frac{d}{d\rho} - \frac{\ell(\ell+1)}{\rho^2}\right] R + R = 0$$

$$\cancel{\frac{d^2u}{d\rho^2} + \frac{2}{\rho}\frac{du}{d\rho} - \frac{\ell(\ell+1)}{\rho^2}u + u = 0}$$

$$\cancel{\ell = 0 \Rightarrow \frac{d^2u}{d\rho^2} + \frac{2}{\rho}\frac{du}{d\rho} + u = 0}$$

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$u = \rho R$ dönüşümü yapalım.

$$\frac{dR}{d\rho} = -\frac{1}{\rho^2}u + \frac{1}{\rho}\frac{du}{d\rho}$$

$$\frac{d^2R}{d\rho^2} = \frac{2}{\rho^3}u - \frac{1}{\rho^2}\frac{du}{d\rho} - \frac{1}{\rho^2}\frac{du}{d\rho} + \frac{1}{\rho}\frac{d^2u}{d\rho^2}$$

$$= \frac{2}{\rho^3}u - \frac{2}{\rho^2}\frac{du}{d\rho} + \frac{1}{\rho}\frac{d^2u}{d\rho^2}$$

$$\cancel{\frac{2}{\rho^3}u} - \cancel{\frac{2}{\rho^2}\frac{du}{d\rho}} + \frac{1}{\rho}\frac{d^2u}{d\rho^2} + \underbrace{\frac{2}{\rho}\frac{d}{d\rho}\left(\frac{u}{\rho}\right)}_{-\cancel{\frac{2}{\rho^3}u} + \cancel{\frac{2}{\rho^2}\frac{du}{dr}}} - \frac{\ell(\ell+1)}{\rho^2}\frac{u}{\rho}$$

$$+ \frac{u}{\rho} = 0$$

$$\boxed{\frac{d^2u}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2}u + u = 0}$$

$$\ell = 0 \Rightarrow \frac{d^2u}{d\rho^2} + u = 0 \;\Big\}\; u(\rho) \sim \begin{cases} \sin\rho \\ \cos\rho \end{cases}$$

$$R \sim \begin{cases} \dfrac{\sin\rho}{\rho} \leftarrow \text{düzgün çözüm} \\[2ex] \dfrac{\cos\rho}{\rho} \leftarrow \text{düzgün olmayan çözüm.} \end{cases}$$

BS K

$$\rho^2 \frac{d^2R}{d\rho^2} + 2\rho \frac{dR}{d\rho} + \left(\rho^2 - \ell(\ell+1)\right) R = 0$$

bunun çözümü küresel Bessel fonk.'larıdır.

$$R(\rho) = \frac{u(\rho)}{\sqrt{\rho}} \quad \text{değişimini yapalım.}$$

$$R' = u' \rho^{-1/2} - \frac{1}{2} \rho^{-3/2} u$$

$$R'' = u'' \rho^{-1/2} - \frac{1}{2} \rho^{-3/2} u' - \frac{1}{2} \rho^{-3/2} u' + \frac{3}{4} \rho^{-5/2} u$$

$$= u'' \rho^{-1/2} - \rho^{-3/2} u' + \frac{3}{4} \rho^{-5/2} u$$

$$\Rightarrow \rho^2 R'' = u'' \rho^{3/2} - u' \rho^{1/2} + \frac{3}{4} u \rho^{-1/2}$$

$$2\rho R' = 2 u' \rho^{1/2} - 2u \rho^{-1/2}$$

+

$$\rho^2 u'' \rho^{-1/2} - \rho^2 u' \rho^{-3/2} + \rho^2 \frac{3}{4} u \rho^{-5/2}$$

$$+ 2\rho u' \rho^{-1/2} + 2\rho \left(-\frac{1}{2}\right) u \rho^{-3/2} + \left(\rho^2 - \ell(\ell+1)\right) u \rho^{-1/2} = 0$$

$$\rho^2 u'' - \rho u' + \frac{3}{4} u + 2\rho u' - u + (\rho^2 - \ell(\ell+1)) u = 0$$

$$-1/4$$

$$\rho^2 u'' + \rho u' + \left[\rho^2 - \ell(\ell+1) - \frac{1}{4}\right] u = 0$$

BS K

$$\rho^2 u'' + \rho u' + \left[\rho^2 - \left(\ell + \tfrac{1}{2}\right)^2\right] u = 0$$

Bunun çözümleri $J_{\ell+1/2}(\rho)$ ve $N_{\ell+1/2}(\rho)$

$$R(\rho) = \frac{c_1}{\sqrt{\rho}} J_{\ell+1/2}(\rho) + \frac{c_2}{\sqrt{\rho}} N_{\ell+1/2}(\rho)$$

$$= c\, j_\ell(\rho) + c' n_\ell(\rho)$$

$$j_\ell(\rho) = \sqrt{\frac{\pi}{2\rho}}\, J_{\ell+\frac{1}{2}}(\rho)$$

$$n_\ell(\rho) = \sqrt{\frac{\pi}{2\rho}}\, N_{\ell+\frac{1}{2}}(\rho)$$

Rekürans bağıntısı: $\dfrac{j_{\ell+1}}{\rho^{\ell+1}} = -\dfrac{1}{\rho}\dfrac{d}{d\rho}\left(\dfrac{j_\ell}{\rho^\ell}\right)$

$\ell = 0$ $\qquad \dfrac{j_1}{\rho} = -\dfrac{1}{\rho}\dfrac{d}{d\rho} j_0$

$\ell = 1$ $\qquad \dfrac{j_2}{\rho^2} = -\dfrac{1}{\rho}\dfrac{d}{d\rho}\left(\dfrac{j_1}{\rho}\right) = -\dfrac{1}{\rho}\dfrac{d}{d\rho}\left(-\dfrac{1}{\rho}\dfrac{d}{d\rho} j_0\right)$

⋮

$\ell - 1$ defa $\qquad j_\ell(\rho) = (-\rho)^\ell \left(\dfrac{1}{\rho}\dfrac{d}{d\rho}\right)^\ell j_0$

BS K $j_0 = \sin\rho / \rho$.

$$j_0(\rho) = \frac{\sin\rho}{\rho}$$

$$j_1(\rho) = \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho}$$

$$j_2(\rho) = \left(\frac{2}{\rho^2} - \frac{1}{\rho}\right)\sin\rho - \frac{3}{\rho^2}\cos\rho$$

$$n_\ell(\rho) = -(-\rho)^\ell \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^\ell \frac{\cos\rho}{\rho}$$

Bunların bileşimleri,

$$h_\ell^{(1)}(\rho) = j_\ell(\rho) + i n_\ell(\rho)$$ Küresel Hankel Fonk.

$$h_\ell^{(2)}(\rho) = j_\ell(\rho) - i n_\ell(\rho)$$

$$h_0^{(1)}(\rho) = \frac{e^{i\rho}}{i\rho} \qquad h_1^{(1)} = -\frac{e^{i\rho}}{\rho}\left(1 + \frac{i}{\rho}\right)$$

$$h_2^{(1)}(\rho) = i\,\frac{e^{i\rho}}{\rho}\left(1 + \frac{3i}{\rho} - \frac{3}{\rho^3}\right)$$

BS K

(a) Orijin civarındaki davranış: $\rho \ll \ell$ için

$$j_\ell(\rho) \approx \frac{\rho^\ell}{1\cdot 3\cdot 5\cdot \ldots \cdot (2\ell+1)}$$

$$n_\ell(\rho) \approx -\frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2\ell-1)}{\rho^{\ell+1}}$$

$\rho \gg \ell$ için $\quad j_\ell(\rho) \simeq \frac{1}{\rho}\sin\left(\rho - \frac{\ell\pi}{2}\right)$

$$R(\rho) = j_\ell(kr)$$

$$\simeq -\frac{1}{2ikr}\left[e^{-i\left(kr - \frac{\ell\pi}{2}\right)} - e^{+i\left(kr - \frac{\ell\pi}{2}\right)} \right]$$

BS K

B. KARE KUTU, BAĞLI DURUMLAR

$$V(r) = \begin{cases} -V_0 & r < a \\ 0 & r > a \end{cases}$$

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{\ell(\ell+1)}{r^2}R + \overbrace{\frac{2\mu}{\hbar^2}(V_0+E)}^{k^2}R = 0 \qquad r<a$$

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{\ell(\ell+1)}{r^2}R + \underbrace{\frac{2\mu E}{\hbar^2}}_{-\alpha^2 = (i\alpha)^2 \text{ alırsak}} R = 0 \qquad r>a$$

$k \to i\alpha$ Bessel'in denk.

$E<0$ olan bağlı durum çözümlerini bulalım.

$$\frac{2\mu}{\hbar^2}(V_0+E) = k^2$$

$$\frac{2\mu}{\hbar^2}E = -\alpha^2$$

$r<a$ için $R(r) = A\, j_\ell(kr)$ başlangıç noktasında düzenli olması için.

$r>a$ çözümü $r \to \infty$ için "0" olmalıdır.

| $k \to i\alpha$ sonsuzda e^{ikr} gibi davranıyordu şimdi $e^{-\alpha r}$ olur.

$$R(r) = B\, h_\ell^{(1)}(i\alpha r)$$

iki çözüm ve türevleri $r=a$'da eşitlenir.

$$\frac{d}{dr} j_\ell(\rho) = k \frac{d}{d\rho} j_\ell(\rho)$$

$$k\left[\frac{dj_\ell(\rho)/d\rho}{j_\ell(\rho)}\right]_{\rho=ka} = i\alpha\left[\frac{dh_\ell^{(1)}(\rho)/d\rho}{h_\ell^{(1)}(\rho)}\right]_{\rho=i\alpha a}$$

ℓ, V_0 ve E için çok karışık bir denklemdir.

$\ell=0$ için basit çözüm:

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + k^2 R = 0 \qquad r<a$$

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \alpha^2 R = 0 \qquad r>a$$

$R(r) = \frac{u(r)}{r}$ dönüşümü yapılır ile

$$\frac{d^2u}{dr^2} + k^2 u = 0 \qquad r<a \qquad u(r) = A\sin kr + \cancel{B\cos kr}$$

$$\frac{d^2u}{dr^2} - \alpha^2 u = 0 \qquad r>a \qquad u(r) = C e^{-\alpha r} + \cancel{D e^{\alpha r}}$$

$r=a$'da $\left.\frac{kA\cos kr}{A\sin kr}\right|_{r=a} = \left.\frac{-\alpha C e^{-\alpha r}}{C e^{-\alpha r}}\right|_{r=a}$

BS K

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$$\eta \cot \eta a = -\alpha \quad / a$$

$$a\eta \cot \eta a = -\alpha a \qquad a\eta = \eta \qquad a\alpha = \xi$$

$$\eta \cot \eta = -\xi$$

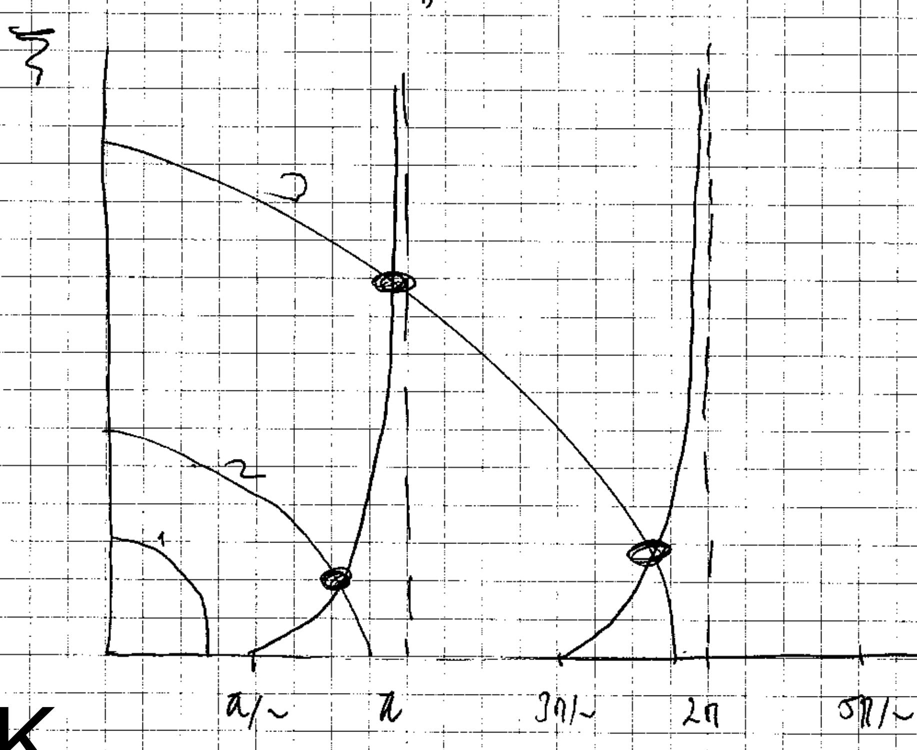
$$\cancel{\eta^2 + \xi^2}$$

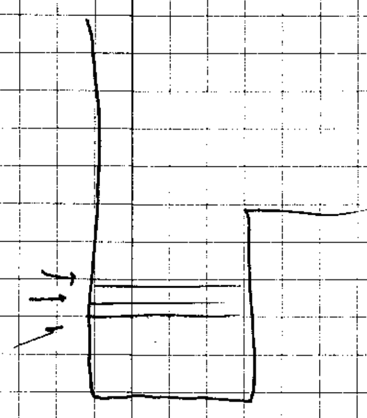
$$\frac{2\mu}{\hbar^2}(V_0 + E) = \eta^2 \qquad \frac{2\mu}{\hbar^2} E = -\alpha^2$$

$$-\alpha^2 + \frac{2\mu}{\hbar^2} V_0 = \eta^2$$

$$\alpha^2 + \eta^2 = \frac{2\mu V_0}{\hbar^2} \quad / a^2$$

$$\eta^2 + \xi^2 = \frac{2\mu V_0}{\hbar^2} a^2 = W^2 \qquad \text{çember denklemi}$$





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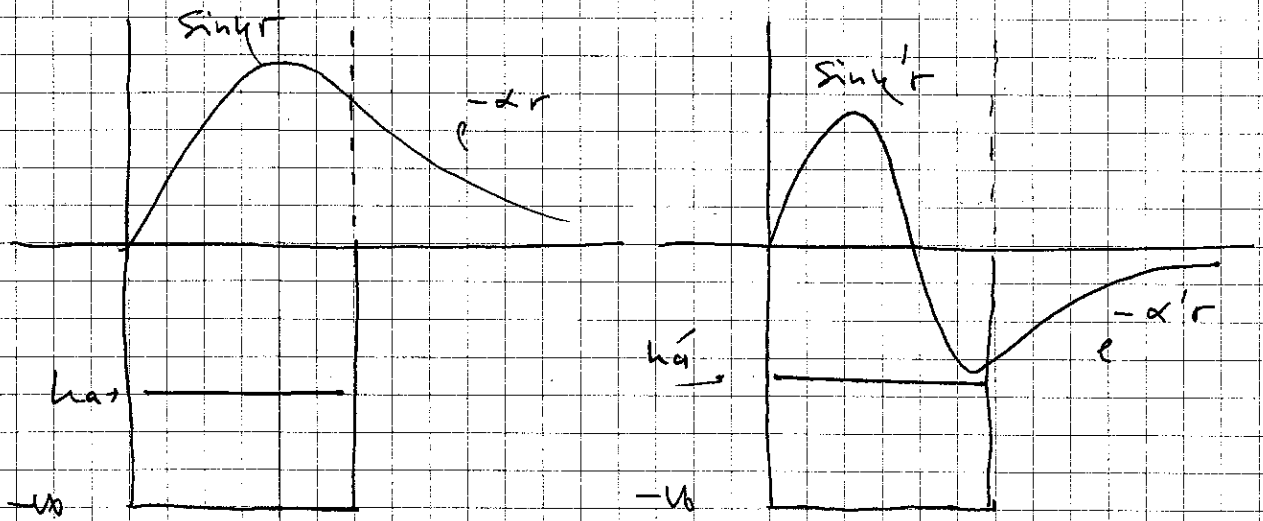


1. bağlı durum. 2. bağlı durum.

Bağlı durum olabilmesi için

$$\frac{2\mu V_0}{\hbar^2} a^2 > \frac{\pi^2}{4} \quad \text{olmalı.}$$

~~C. Kare Kuyu~~: SÜREKLİ ÇÖZÜMLER

$E > 0$ olduğunda $\frac{2\mu E}{\hbar^2} = k^2$

$r < a$ için $R_\ell(r) = A\, j_\ell(kr)$ için

$r > a$ " $R_\ell(r) = B\, j_\ell(kr) + C\, n_\ell(kr)$ olur.

$$K^2 = \frac{2\mu}{\hbar^2}(E + V_0)$$

$r = a$'da $$K\left[\frac{d j_\ell(\rho)/d\rho}{j_\ell(\rho)}\right]_{\rho = Ka} = k\left[\frac{B\, dj_\ell/d\rho + C\, dn_\ell/d\rho}{B\, j_\ell(\rho) + C\, n_\ell(\rho)}\right]_{\rho = ka}$$

BS K

$\frac{C}{B}$ oranını faz kayması olarak yorumlayacağız. $j_\ell(\rho)$ ve $n_\ell(\rho)$'nun asimtotik değerlerine bakarız ($r>a$ için)

$$R_\ell(r) = \frac{B}{kr}\left[\sin\left(kr - \frac{\ell\pi}{2}\right) - \frac{C}{B}\cos\left(kr - \frac{\ell\pi}{2}\right)\right]$$

$\frac{C}{B} = -\tan\delta_\ell(k)$ dersek,

$$R_\ell(r) = \frac{B}{kr}\frac{1}{\cos\delta_\ell(k)}\left[\sin\left(kr - \frac{\ell\pi}{2}\right)\cos\delta_\ell(k) + \cos\left(kr - \frac{\ell\pi}{2}\right)\sin\delta_\ell(k)\right]$$

$$= \frac{C'}{r}\sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell(k)\right)$$

$\ell = 0$ için $u(r) = rR(r)$ değişimi yapılırsa

$$u(r) = A\sin\eta r \qquad r<a$$

$$u(r) = B\sin kr + C\cos kr \qquad r>a$$

$$\eta\frac{\cos\eta a}{\sin\eta a} = k\frac{B\cos ka - C\sin ka}{B\sin ka + C\cos ka}$$

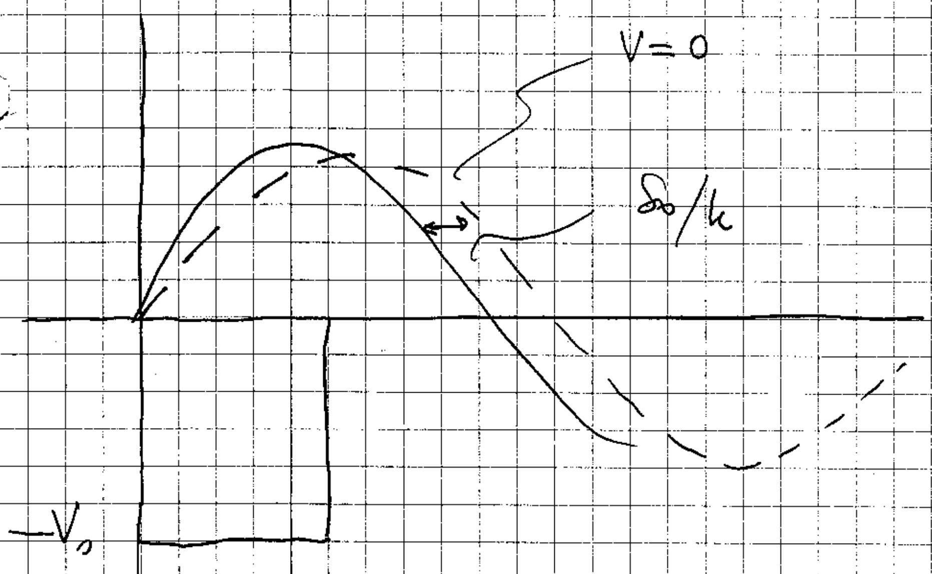
$$\eta\cot\eta a = k\frac{\cos ka - \frac{C}{B}\sin ka}{\sin ka + \frac{C}{B}\cos ka} \qquad \frac{C}{B} = \mathrm{tg}\,\delta_0$$

BS K

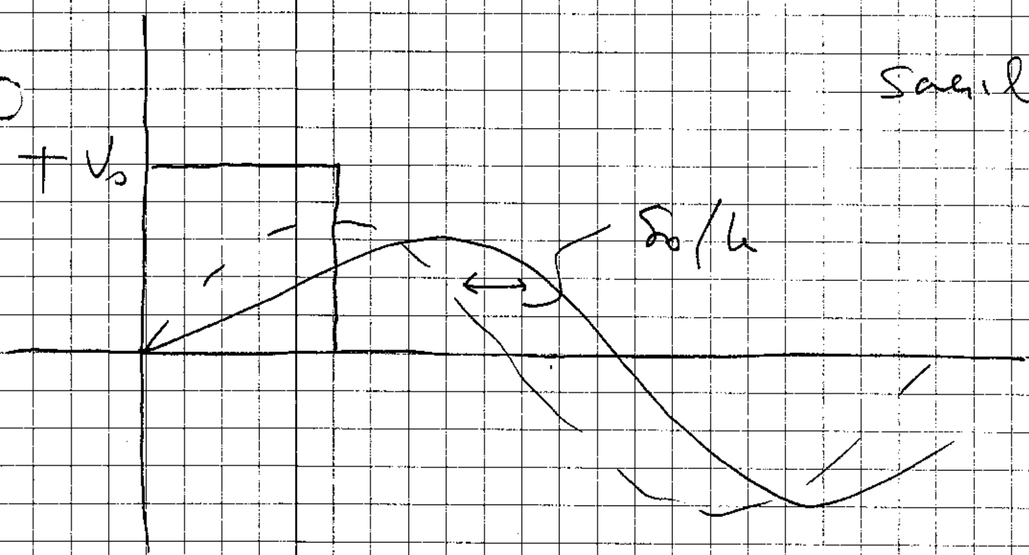
$$\kappa \coth \kappa a = k \frac{\cos(ka+\delta_0)}{\sin(ka+\delta_0)} = k \cot(ka+\delta_0)$$

$$\tan(ka+\delta_0) = \frac{k}{\kappa} \tan \kappa a$$

$$\delta_0 = -ka + n\pi + \mathrm{Arctan}\left(\frac{k}{\kappa} \tan \kappa a\right)$$



$\ell = 0$



Saçılma prob. ile geri dönülecek.

D. Kutu içindeki Parçacık:

$$V(r) = \begin{cases} 0 & r < R \\ \infty & r > R \end{cases} \qquad k^2 = 2\mu E/\hbar^2$$

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{\ell(\ell+1)}{r^2}R + \frac{2\mu E}{\hbar^2}R = 0 \qquad r < R$$

k^2

$$R(r) = 0 \qquad r > R$$

$$u = rR$$

$$\frac{dR}{dr} = -\frac{1}{r^2}u + \frac{1}{r}\frac{du}{dr} \qquad \frac{d^2R}{dr^2} = \frac{2}{r^3}u - \frac{1}{r^2}u' - \frac{1}{r^2}u' + \frac{1}{r}\frac{d^2u}{dr^2}$$

$$= \frac{2}{r^3}u - \frac{2}{r^2}u' + \frac{1}{r}u''$$

$$\cancel{\frac{2}{r^3}u} - \cancel{\frac{2}{r^2}u'} + \frac{1}{r}u'' + \frac{2}{r}\left(\cancel{-\frac{1}{r^2}u} + \cancel{\frac{1}{r}u'}\right) - \frac{\ell(\ell+1)}{r^3}u + k^2\frac{u}{r}$$

$$\frac{1}{r}u'' - \frac{\ell(\ell+1)}{r^3}u + k^2\frac{u}{r} = 0$$

$$\boxed{u'' + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right)u = 0}$$

2.) $\quad u = z^{1/2}\varphi(z) \qquad$ 1. $\quad z = kr$

BS K

$$\frac{d}{dr} = \frac{dz}{dr}\frac{d}{dz} = k\frac{d}{dz} \qquad \frac{d^2}{dr^2} = k^2\frac{d^2}{dz^2}$$

$$k^2\frac{d^2u}{dz^2} + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right)u = 0$$

$$\frac{d^2u}{dz^2} + \left(1 - \frac{\ell(\ell+1)}{z^2}\right)u = 0$$

$$\frac{du}{dz} = \frac{1}{2}z^{-1/2}\varphi + z^{1/2}\varphi'$$

$$\frac{d^2u}{dz^2} = -\frac{1}{4}z^{-3/2}\varphi + \frac{1}{2}z^{-1/2}\varphi' + \frac{1}{2}z^{-1/2}\varphi' + z^{1/2}\varphi''$$

$$= z^{1/2}\varphi'' + z^{-1/2}\varphi' - \frac{1}{4}z^{-3/2}\varphi$$

$$z^{1/2}\varphi'' + z^{-1/2}\varphi' - \frac{1}{4}z^{-3/2}\varphi + \left[1 - \frac{\ell(\ell+1)}{z^2}\right]z^{\frac{1}{2}}\varphi = 0$$

$$\varphi'' + \frac{1}{z}\varphi' + \left[\underbrace{-\frac{1}{4z^2} + 1 - \frac{\ell(\ell+1)}{z^2}}\right]\varphi = 0$$

$$1 - \frac{\ell(\ell+1) + 1/4}{z^2} = 1 - \frac{\ell^2 + \ell + 1/4}{z^2}$$

$$= 1 - \frac{\left(\ell + \frac{1}{2}\right)^2}{z^2}$$

$$\varphi'' + \frac{1}{z}\varphi' + \left[1 - \frac{\left(\ell + \frac{1}{2}\right)^2}{z^2}\right]\varphi = 0$$

BS K

$$\varphi_\ell(z) = C_1 J_{\ell+\frac{1}{2}}(z) + C_2 J_{-(\ell+1/2)}(z)$$

$$j_\ell(z) = \sqrt{\frac{\pi z}{2}}\, J_{\ell+1/2}(z)$$

$$n_\ell(z) = (-1)^{\ell+1} \sqrt{\frac{\pi z}{2}}\, J_{-\ell+1/2}(z)$$

$$u(z) = z^{1/2}\left[C_1 J_{\ell+1/2} + C_2 J_{-(\ell+1/2)}\right]$$

$$= C_1' j_\ell + C_2' n_\ell$$

alınmamalı.

$$R(r) = \frac{u(kr)}{kr} = \frac{1}{r}\left(C_1'' j_\ell(kr) + \cancel{C_2'' n_\ell(kr)}\right)$$

$r=0$ da $r^{-\ell}$ gibi davranır

$$R(r) = \frac{C_1''}{r} j_\ell(kr)$$

$r=R$ de $R(r=R)=0 \Rightarrow j_\ell(kR)=0$

$$\Rightarrow J_{\ell+\frac{1}{2}}(kR)=0$$

kökleri bulunmalı.

BS K

Her $\ell+\frac{1}{2}$ için Bessel fonk.'un sonsuz sayıda sıfırlara sahip.

Her bir ℓ için sıfırları sayan $n_r=1,2,3,\ldots$ radyal kuantum sayısı ile birlikte enerji

$$E_{n_r\ell}=\frac{\hbar^2}{2m}\,k^2_{n_r,\ell}$$

$$j_0(z)=\sin z \text{ idi} \qquad j_1(z)=\frac{\sin z}{z}-\cos z$$

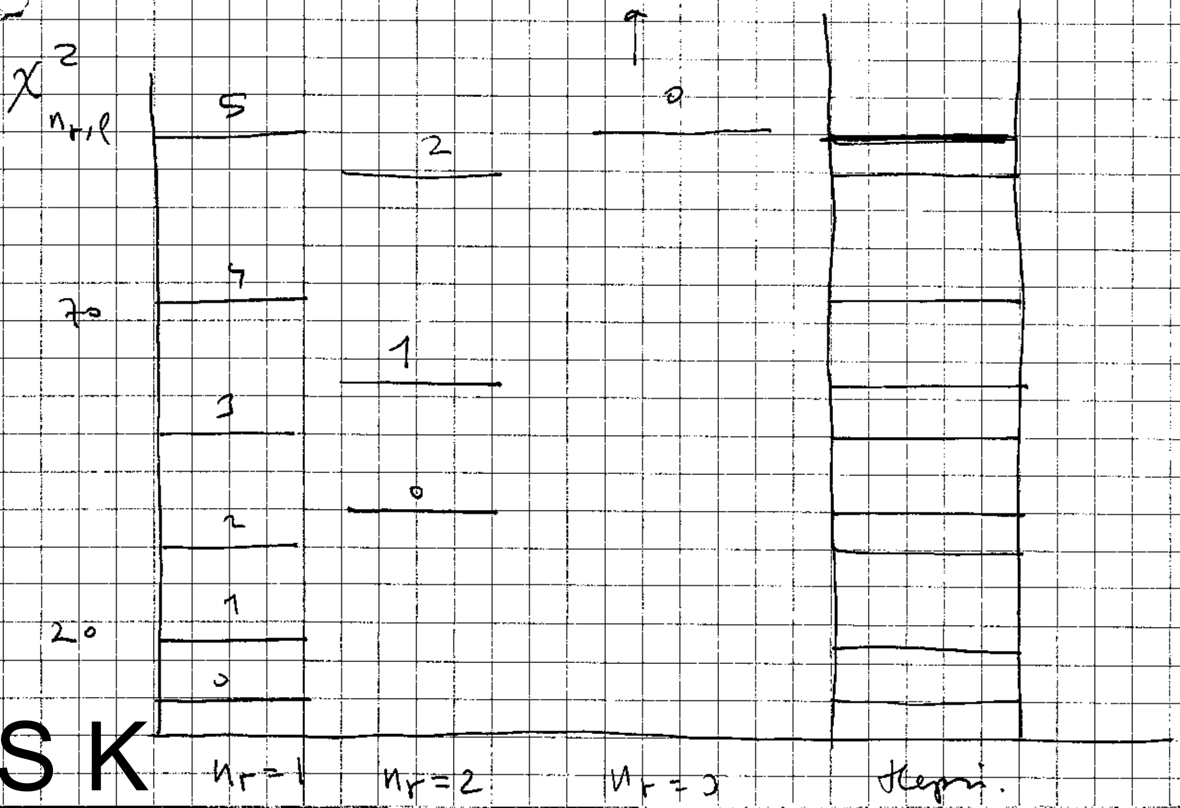
$$j_0(z)=0 \Rightarrow \sin z=0 \Rightarrow z=n_r\pi$$

$$j_1(z)=0 \Rightarrow \tan z=z,\ \ldots$$

Hepsi çift ℓ'ler için $n\pi$

tek ℓ'ler için $(\ell+1/2)$'ye gider.

$\chi_{n_r,\ell}=k_{n_r\ell}R$ dersek



l \ n_r	1	2	3	4
0	3.142	6.283	9.425	12.566
1	4.493	7.725	10.904	
2	5.764	9.095	12.323	
3	6.988	10.417		

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$$H = \frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_2} + V(\vec{r}_1, \vec{r}_2) \qquad V(\vec{r}_1, \vec{r}_2) = V(|\vec{r}_1 - \vec{r}_2|) \text{ olsun}$$

$\vec{r}_1 \to \vec{r}_1 + \vec{a}$ $\quad \vec{r}_2 \to \vec{r}_2 + \vec{a}$ yerdeğiştirmeleri altında değişmez.

$$-\hbar^2 \left(\frac{1}{2m_1} \nabla_1^2 + \frac{1}{2m_2} \nabla_2^2 \right) \psi(\vec{r}_1, \vec{r}_2) + V(|\vec{r}_1 - \vec{r}_2|)\psi(\vec{r}_1, \vec{r}_2) = E\psi(\vec{r}_1, \vec{r}_2)$$

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \qquad \vec{r} = \vec{r}_1 - \vec{r}_2$$

$$\cancel{R_i = m_1 r_2} \quad X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \quad Y = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}, \quad Z = \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2}$$

$x_1(x, X)$ $\quad x_2(x, X)$ koordinat dönüşümlerini yapalım.

$(x_1 - x_2)$

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial X}\frac{\partial X}{\partial x_1} + \frac{\partial}{\partial x}\frac{\partial x}{\partial x_1} = \frac{m_1}{m_1+m_2}\frac{\partial}{\partial X} + \frac{\partial}{\partial x}$$

$$\frac{\partial^2}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left[\frac{m_1}{m_1+m_2}\frac{\partial}{\partial X} + \frac{\partial}{\partial x}\right] = \left[\frac{m_1}{m_1+m_2}\frac{\partial}{\partial X} + \frac{\partial}{\partial x}\right]\left[\frac{m_1}{m_1+m_2}\frac{\partial}{\partial X} + \frac{\partial}{\partial x}\right]$$

$$= \left(\frac{m_1}{m_1+m_2}\right)^2 \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial x^2} + \cancel{2\frac{m_1}{m_1+m_2}\frac{\partial^2}{\partial X \partial x}} \qquad \times \frac{1}{2m_1}$$

$$\frac{\partial^2}{\partial x_2^2} = \left(\frac{m_2}{m_1+m_2}\right)^2 \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial x^2} - \cancel{2\frac{m_2}{m_1+m_2}\frac{\partial^2}{\partial X \partial x}} \qquad \times \frac{1}{2m_2}$$

$$\frac{1}{2m_1}\frac{\partial^2}{\partial x_1^2} + \frac{1}{2m_2}\frac{\partial^2}{\partial x_2^2} = \frac{\cancel{(m_1+m_2)}}{2(m_1+m_2)^2}\frac{\partial^2}{\partial X^2} + \frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\frac{\partial^2}{\partial x^2} = \frac{1}{2M}\frac{\partial^2}{\partial X^2} + \frac{1}{2\mu}\frac{\partial^2}{\partial x^2}$$

6.

$$\vec{P}^2 \to -\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+\frac{\partial^2}{\partial Z^2}\right) - \frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)$$

$$\left[-\hbar^2\left(\frac{1}{2M}\vec{\nabla}_R^2+\frac{1}{2\mu}\vec{\nabla}_r^2\right)+V(\vec{r})\right]\psi(\vec{R},\vec{r}) = E\,\psi(\vec{R},\vec{r})$$

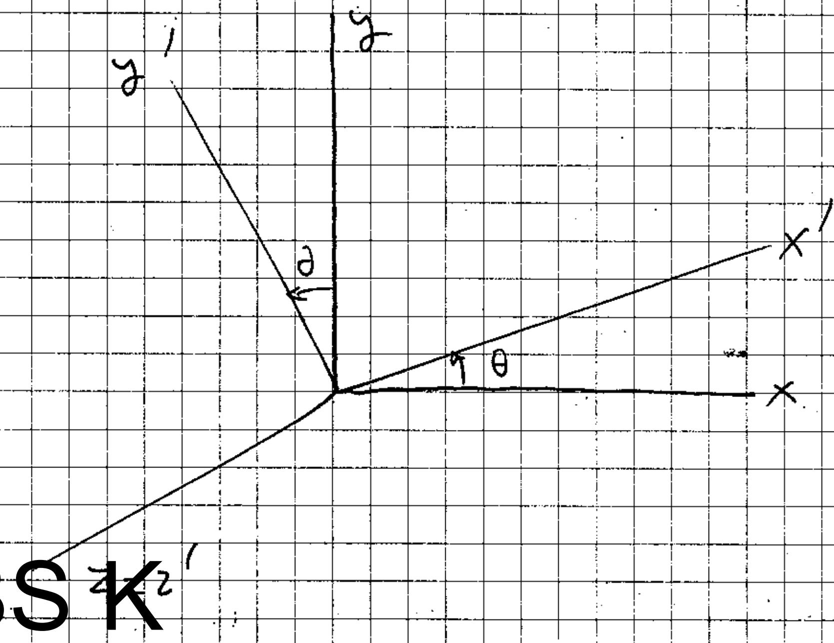
$$\psi(\vec{R},\vec{r}) = \Phi(\vec{R})\,\phi(\vec{r})$$

$$\boxed{-\frac{\hbar^2}{2M}\vec{\nabla}_R^2\,\Phi(\vec{R}) = E_1\,\Phi(\vec{R})}$$

$$\Rightarrow \psi(\vec{R},\vec{r}) = K\,\phi(\vec{r})\,e^{i\vec{k}\cdot\vec{R}}$$

$$\boxed{\left[-\frac{\hbar^2}{2\mu}\vec{\nabla}_r^2+V(\vec{r})\right]\phi(\vec{r}) = E_2\,\phi(\vec{r})}$$

Hamiltoniyeni dönmeler altında değişmezdir. Bunu görmek için z-ekseni etrafındaki dönmeyi ele alalım.



$$x' = x\cos\theta + y\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta$$

$$\underset{\sim}{r}' = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\underset{\sim}{r}$$

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'} = \cos\theta\frac{\partial}{\partial x'} - \sin\theta\frac{\partial}{\partial y'}$$

$$\frac{\partial^2}{\partial x^2} = \cos\theta\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x'}\right) - \sin\theta\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y'}\right)$$

$$= \cos\theta\left(\cos\theta\frac{\partial}{\partial x'} - \sin\theta\frac{\partial}{\partial y'}\right)\frac{\partial}{\partial x'} - \sin\theta\left(\cos\theta\frac{\partial}{\partial x'} - \sin\theta\frac{\partial}{\partial y'}\right)\frac{\partial}{\partial y'}$$

$$= \cos^2\theta\frac{\partial^2}{\partial x'^2} - \cancel{2\cos\theta\sin\theta\frac{\partial^2}{\partial x'\partial y'}} + \sin^2\theta\frac{\partial^2}{\partial y'^2}$$

$$\frac{\partial^2}{\partial y^2} = \sin^2\theta\frac{\partial^2}{\partial x'^2} + \cancel{2\cos\theta\sin\theta\frac{\partial^2}{\partial x'\partial y'}} + \cos^2\theta\frac{\partial^2}{\partial y'^2}$$

$\nabla^2 = \nabla'^2$ dönmeler altında değişmez $\vec{\partial}$ ler de dönmeden etkilenmeyen nicelikler. sonsuz küçük dönmeler altında

$$\begin{pmatrix} x' \simeq x + y\theta \\ y' \simeq y - x\theta \end{pmatrix} \qquad \begin{matrix} \cos\theta \simeq 1 \\ \sin\theta \simeq \theta \end{matrix}$$

$$\psi(x+y\theta,\, y-x\theta,\, z) = \psi(x,y,z) + y\theta\frac{\partial\psi}{\partial x} - x\theta\frac{\partial\psi}{\partial y} + \dots$$

$$H\left[\psi(x,y,z) + y\theta\frac{\partial\psi}{\partial x} - x\theta\frac{\partial\psi}{\partial y} + \dots\right] = E\left[\psi(x,y,z) + y\theta\frac{\partial\psi}{\partial x} - x\theta\frac{\partial\psi}{\partial y} + \dots\right]$$

$$H\psi(x,y,z) = E\psi(x,y,z)$$

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sbt.

$$H\left[y\theta\frac{\partial}{\partial x} - x\theta\frac{\partial}{\partial y}\right]\psi = \left[y\theta\frac{\partial}{\partial x} - x\theta\frac{\partial}{\partial y}\right]E\psi$$

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$$\frac{\hbar}{i} H \left[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \right] \psi = \frac{\hbar}{i} \left[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \right] H\psi$$

$$H(xp_y - yp_x)\psi = (xp_y - yp_x)H\psi$$

$$HL_z - L_zH = [H, L_z] = 0$$

L_z komuter.

x ve y etrafındaki dönmelere de bakılsa idi,

$$[H, L_x] = 0$$

$$[H, L_y] = 0$$

bulurduk

$$[H, \vec{L}] = 0$$

$$[H, L_x^2] = L_x[H, L_x] + [H, L_x]L_x = 0$$

$$\Rightarrow [H, \vec{L}^2] = 0$$

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