BÖLÜM 12

AÇISAL MOMENTUM

$$-\frac{\hbar^2}{2m}\vec{\nabla}^2 = -\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\vec{L}^2}{r^2\hbar^2}\right]$$

$$\psi \equiv \psi(r,\theta,\varphi) = R_{n\ell}(r)\, Y_{\ell m}(\theta,\varphi)$$

$$\vec{L}^2\, Y_{\ell m}(\theta,\varphi) = \ell(\ell+1)\hbar^2\, Y_{\ell m}(\theta,\varphi)$$

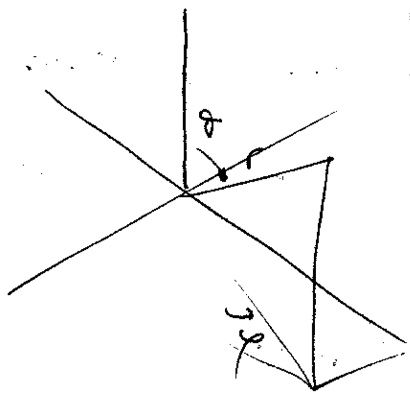
L_z ve $\vec{L}^2$ eşzamanlı ortak özfonk. lara sahip

$$L_z\, Y_{\ell m}(\theta,\varphi) = m\hbar\, Y_{\ell m}(\theta,\varphi)$$

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$



$$dx = \sin\theta\cos\varphi\, dr + r\cos\theta\cos\varphi\, d\theta - r\sin\theta\sin\varphi\, d\varphi$$

$$dy = \sin\theta\sin\varphi\, dr + r\cos\theta\sin\varphi\, d\theta + r\sin\theta\cos\varphi\, d\varphi$$

$$dz = \cos\theta\, dr - r\sin\theta\, d\theta + 0\cdot d\varphi$$

BS K

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix} \begin{bmatrix} dr \\ d\theta \\ d\varphi \end{bmatrix}$$

ortogonal

$$= \underset{\sim}{A} \begin{pmatrix} dr \\ d\theta \\ d\varphi \end{pmatrix}$$

$$\Rightarrow \underset{\sim}{A}^{-1} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \underbrace{\underset{\sim}{A}^{-1} \underset{\sim}{A}}_{I} \begin{pmatrix} dr \\ d\theta \\ d\varphi \end{pmatrix}$$

$$A^{-1} = \frac{(A^c)'}{\det A}$$

$$\det A = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$

$(A^c)' = (A')^c$ olduğundan

$$A' = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ r\cos\theta\cos\varphi & r\cos\theta\sin\varphi & -r\sin\theta \\ -r\sin\theta\sin\varphi & r\sin\theta\cos\varphi & 0 \end{bmatrix}$$

BS K

$$\left(A'\right)^{c} = \begin{bmatrix} (-1)^{1+1}\,|A'^{11}| & (-1)^{1+2}\,|A'^{12}| & (-1)^{1+3}\,|A'^{13}| \\ (-1)^{2+1}\,|A'^{21}| & (-1)^{2+2}\,|A'^{22}| & (-1)^{2+3}\,|A'^{23}| \\ (-1)^{3+1}\,|A'^{31}| & (-1)^{3+2}\,|A'^{32}| & (-1)^{3+3}\,|A'^{33}| \end{bmatrix}$$

$$|A'^{11}| = \begin{vmatrix} r\cos\theta\sin\varphi & -r\sin\theta \\ r\sin\theta\cos\varphi & 0 \end{vmatrix} = r^2\sin^2\theta\cos\varphi$$

$$|A'^{12}| = \begin{vmatrix} r\cos\theta\cos\varphi & -r\sin\theta \\ -r\sin\theta\sin\varphi & 0 \end{vmatrix} = -r^2\sin^2\theta\sin\varphi$$

$$|A'^{13}| = r^2\sin\theta\cos\theta \qquad |A'|^{21} = -r\sin\theta\cos\theta\cos\varphi$$

$$|A'^{21}| = -r\sin\theta\cos\theta\cos\varphi \;\ldots$$

$$\left(A'\right)^{c} = \begin{pmatrix} r^2\sin^2\theta\cos\varphi & r^2\sin^2\theta\sin\varphi & r^2\sin\theta\cos\theta \\ r\sin\theta\cos\theta\cos\varphi & r\sin\theta\cos\theta\cos\varphi & -r\sin^2\theta \\ -r\sin\varphi & r\cos\varphi & 0 \end{pmatrix}$$

$$A^{-1} = = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi/r & \cos\theta\sin\varphi/r & -\sin\theta/r \\ -\sin\varphi/r\sin\theta & \cos\varphi/r\sin\theta & 0 \end{pmatrix}$$

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$$dr = \sin\theta\cos\varphi\, dx + \sin\theta\sin\varphi\, dy + \cos\theta\, dz$$

$$d\theta = \frac{1}{r}\cos\theta\cos\varphi\, dx + \frac{1}{r}\cos\theta\sin\varphi\, dy - \frac{1}{r}\sin\theta\, dz$$

$$d\varphi = -\frac{1}{r}\frac{\sin\varphi}{\sin\theta}\, dx + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\, dy + 0 \cdot dz.$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi}$$

$$= \sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}$$

$$L_z = x p_y - y p_x = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$= \frac{\hbar}{i}\Big\{ r\sin\theta\cos\varphi\Big[\cancel{\sin\theta\sin\varphi\frac{\partial}{\partial r}} + \cancel{\frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta}} + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\Big]$$
$$- r\sin\theta\sin\varphi\Big[\cancel{\sin\theta\cos\varphi\frac{\partial}{\partial r}} + \cancel{\frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta}} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\Big]\Big\}$$

$$= \frac{\hbar}{i}\left(\cos^2\varphi + \sin^2\varphi\right)\frac{\partial}{\partial\varphi} = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}$$

BS K

$$L_x = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$= \frac{\hbar}{i}\left\{\cancel{r}\sin\theta\sin\varphi\left(\cancel{\cos\theta\frac{\partial}{\partial r}} - \frac{1}{\cancel{r}}\sin\theta\frac{\partial}{\partial\theta}\right)\right.$$

$$\left. - \cancel{r}\cos\theta\left(\cancel{\sin\theta\sin\varphi\frac{\partial}{\partial r}} + \frac{1}{\cancel{r}}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{1}{\cancel{r}}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\right\}$$

$$= \frac{\hbar}{i}\left[-\sin^2\theta\sin\varphi\frac{\partial}{\partial\theta} - \cos^2\theta\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right]$$

$$\boxed{L_x = -\frac{\hbar}{i}\left[\sin\varphi\frac{\partial}{\partial\theta} + \cos\varphi\cot\theta\frac{\partial}{\partial\varphi}\right]}$$

$$L_y = \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$= \frac{\hbar}{i}\left\{\cancel{r}\cos\theta\left(\cancel{\sin\theta\cos\varphi\frac{\partial}{\partial r}} + \frac{1}{\cancel{r}}\cos\theta\cos\varphi\frac{\partial}{\partial\theta}\right) - \frac{1}{\cancel{r}}\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$\left. - \cancel{r}\sin\theta\cos\varphi\left(\cancel{\cos\theta\frac{\partial}{\partial r}} - \frac{1}{\cancel{r}}\sin\theta\frac{\partial}{\partial\theta}\right)\right\}$$

$$= \frac{\hbar}{i}\left[\cos^2\theta\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi} + \sin^2\theta \quad \varphi\frac{\partial}{\partial\theta}\right]$$

$$\boxed{L_y = \frac{\hbar}{i}\left[\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right]}$$

$$L_{\pm} = L_x \pm iL_y$$

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$$L_{\pm} = i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right) \pm \hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$$

$$= \left(\pm\hbar\cos\varphi\frac{\partial}{\partial\theta} + i\hbar\sin\varphi\frac{\partial}{\partial\theta}\right)$$

$$+ \left(\mp\hbar\cot\theta\sin\varphi\frac{\partial}{\partial\varphi} + i\cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$$

$$= \pm\hbar\left(\cos\varphi \pm i\sin\varphi\right)\frac{\partial}{\partial\theta}$$

$$+ i\cot\theta\left(\underbrace{\cos\varphi \mp \frac{1}{i}\sin\varphi}_{(\cos\varphi \pm i\sin\varphi)}\right)\hbar\frac{\partial}{\partial\varphi}$$

$$= \pm\hbar\, e^{\pm i\varphi}\frac{\partial}{\partial\theta} + i\cot\theta\,\hbar\, e^{\pm i\varphi}\frac{\partial}{\partial\varphi}$$

$$= \hbar\, e^{\pm i\varphi}\left[\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right]$$

Şimdi $\vec{L}^2$ op.'ünü kuralım.

$$L_+L_- = (L_x + iL_y)(L_x - iL_y)$$

$$= L_x^2 + L_y^2 - iL_xL_y + iL_yL_x$$

$$= L_x^2 + L_y^2 - i\left(L_xL_y - L_yL_x\right)$$

$$= L_x^2 + L_y^2 - i\underbrace{[L_x, L_y]}_{i\hbar L_z}$$

$$= L_x^2 + L_y^2 + L_z$$

$$\Rightarrow \vec{L}^2 = L_x^2 + L_y^2 + L_z^2 = L_z^2 + L_+L_- - \hbar L_z$$

L_z'nin Özdeğer ve Özfonksiyonları:

$$L_z Y_{\ell m} = m\hbar Y_{\ell m} \qquad L_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}$$

denklemini gözönüne alalım.

$$\frac{\partial}{\partial\varphi} Y_{\ell m} = im Y_{\ell m} \qquad Y_{\ell m} = Y_{\ell m}(\theta,\varphi)$$

$$Y_{\ell m}(\theta,\varphi) = \Theta_{\ell m}(\theta)\,\Phi_m(\varphi)$$

$$\Rightarrow \qquad \frac{d\Phi_m}{d\varphi} = im\Phi(\varphi) \Rightarrow \Phi_m(\varphi) = A\,e^{im\varphi}$$

$$\int_0^{2\pi} d\varphi\, |\Phi_m(\varphi)|^2 = 1 \Rightarrow A = 1/\sqrt{2\pi}$$

$$\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}}\,e^{im\varphi}$$

2π'lik dönmeler sist.'i invaryant bırakır. Bu yüzden

$$e^{2\pi i m} = 1 \qquad *(\varphi \to \varphi + 2\pi.)$$

ve m'nin de tamsayı olması gerekir. Ama bu doğru değildir. Çünkü fiziksel gözlenebilirlere giden büyüklükler

$$\int_0^{2\pi} d\phi\, \psi_1^*(\phi)\, A\, \psi_2(\phi)$$

şeklindedir. Burada

$$\psi(\phi) = \sum_{m=-\infty}^{+\infty} C_m \frac{e^{im\phi}}{\sqrt{2\pi}}$$ 'dır.

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Bu dalga paketinin $\phi \to \phi + 2\pi$ dönüşümü altında bir faz çarpanı dışında invaryant kalmasını istersek,

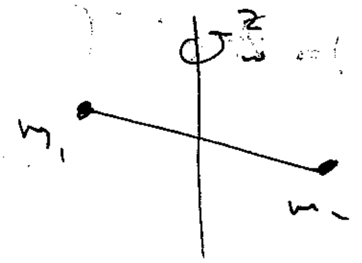
$$m = c + \text{tamsayı}$$

olmalıdır. $c = 0, 1/2$ olabilir. $c = 0$ durumunu inceleyeceğiz.

Rotatör.

$$E = \frac{\ell_z^2}{2I}$$

$$\downarrow$$



$$H = \frac{\hat{L}_z^2}{2I} \Rightarrow -\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\varphi^2}\Phi_m(\varphi) = E_m \Phi_m(\varphi)$$

$$\frac{d^2\Phi_m}{d\varphi^2} + \underbrace{\frac{2IE_m}{\hbar^2}}_{m^2}\Phi_m = 0 \Rightarrow \Phi_m \sim e^{\pm im\varphi}$$

$$\frac{2IE_m}{\hbar^2} = m^2$$

$[H, L_z] = 0$ olduğundan dejenerelik vardır.

Verilen bir E_m için iki özfonk. dönmenin iki yönüne karşı gelir.

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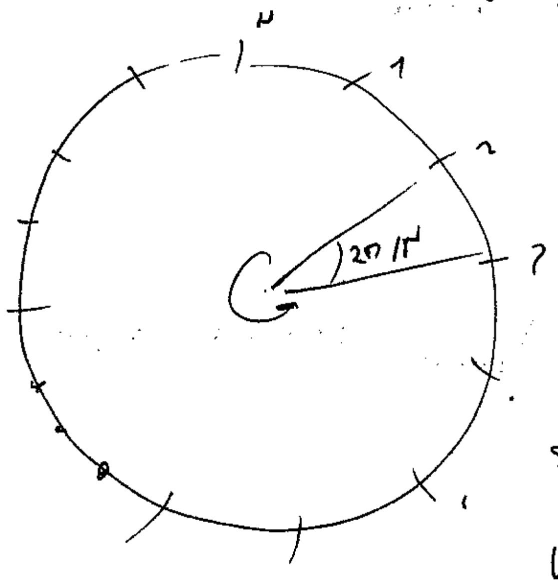
Eğer bir çember üzerinde, aralarında $2\pi/N$ açısı olan N özdeş parçacığımız varsa



$$H\Phi_E = E\Phi_E \qquad \Phi_E \equiv \Phi_E(\varphi)$$

$$\Phi_E(\varphi) \sim e^{\pm i\lambda\varphi}$$

şart: $\frac{2\pi}{N}$ rad'lık ya da ~~tam~~ tam katı kadarlık açılar altında invaryant

$$E = \frac{\hbar^2 (Nm)^2}{2I}$$

BS K

Şimdi

$$\vec{L}^2 Y_{\ell m} = \ell(\ell+1)\hbar^2 Y_{\ell m}$$

denklemini gözönüne alalım.

$$\langle Y_{\ell' m'} | Y_{\ell m} \rangle = \delta_{\ell\ell'} \delta_{mm'}$$ normalizeler.

$$\Rightarrow \langle Y_{\ell m} | (L_x^2 + L_y^2 + L_z^2) Y_{\ell m} \rangle =$$

$$= \langle L_x Y_{\ell m} | L_x Y_{\ell m} \rangle + \langle L_y Y_{\ell m} | L_y Y_{\ell m} \rangle + m^2 \hbar^2$$

≥ 0 olmalı.

$\Rightarrow \ell(\ell+1) \geq 0$ olmalı.

$L_{\pm} = L_x \pm i L_y$ için.

$$L_+ L_- = L_x^2 + L_y^2 + L_z \hbar$$

$$L_- L_+ = L_x^2 + L_y^2 - L_z \hbar$$

$$\Rightarrow \begin{cases} \vec{L}^2 = L_+ L_- + L_z^2 - \hbar L_z \\ \vec{L}^2 = L_- L_+ + L_z^2 + \hbar L_z \end{cases}$$

BS K $L_+ L_- - L_- L_+ = 2\hbar L_z = [L_+, L_-]$

$$[L_-, L_+] = -2\hbar L_z$$

$$[L_+, L_z] = [L_x + iL_y, L_z]$$

$$= [L_x, L_z] + i[L_y, L_z]$$

$$= -i\hbar L_y + i^2\hbar L_x$$

$$= -\hbar(L_x + iL_y) = -\hbar L_+$$

$$[L_-, L_z] = \hbar L_-$$

$$[L^2, L_\pm] = 0 \qquad [\vec{L}^2, L_z] = 0$$

$$\Rightarrow \vec{L}^2 \underline{L_\pm Y_{\ell m}} = L_\pm \vec{L}^2 Y_{\ell m}$$

$$= \hbar^2 \ell(\ell+1) \underline{L_\pm Y_{\ell m}(\vartheta, \varphi)}$$

$$[L_+, L_z] = -\hbar L_+$$

$$\Rightarrow L_+ L_z Y_{\ell m} = L_z L_+ Y_{\ell m} - \hbar L_+ Y_{\ell m}$$

$$\hbar m L_+ Y_{\ell m} + \hbar L_+ Y_{\ell m} = L_z L_+ Y_{\ell m}$$

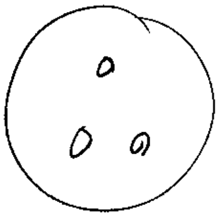
$$\hbar(m+1) \underline{\underline{L_+ Y_{\ell m}}} = L_z \underline{\underline{L_+ Y_{\ell m}}}$$

$L_+ Y_{\ell m}$: L_z'in özfonk.

L_+ : arttırıcı operatör.

$L_z \underline{\underline{L_- Y_{\ell m}}} = \hbar (m-1) \underline{\underline{L_- Y_{\ell m}}}$ bu da L_z'nin özfonk.u..

L_- : azaltıcı operatör.



$$L_\pm Y_{\ell m}(\theta,\varphi) = C_\pm(\ell,m)\, Y_{\ell,m\pm1}(\theta,\varphi)$$

L_x ve L_y hermitik idi : $L_x = L_x^\dagger$, $L_y = L_y^\dagger$

$$L_\pm^\dagger = (L_x \pm i L_y)^\dagger = (L_x^\dagger \mp i L_y^\dagger)$$

$$= L_\mp$$

$L_\mp$'nin karesinin beklenen değeri pozitif'tir.

$$\langle L_\pm Y_{\ell m} | L_\pm Y_{\ell m} \rangle \geq 0$$

$$\langle Y_{\ell m} | L_\mp L_\pm Y_{\ell m} \rangle \geq 0$$

$$L_+ L_- = \vec{L}^2 - L_z^2 + \hbar L_z$$

$$L_- L_+ = \vec{L}^2 - L_z^2 - \hbar L_z$$

$$\Rightarrow \langle Y_{lm} | (\vec{L}^2 - L_z^2 \mp \hbar L_z) Y_{lm} \rangle \geq 0$$

$$l(l+1)\hbar^2 - m^2\hbar^2 \mp m\hbar^2 \geq 0$$

$$l(l+1) - (m^2 \pm m) \geq 0$$

$$l(l+1) \geq m(m \pm 1)$$

$$\Rightarrow \begin{cases} l(l+1) \geq m(m+1) \\ l(l+1) \geq m(m-1) \end{cases}$$

$l(l+1) \geq 0$ 'da vardı.

m 'nin en küçük değeri m_-
" " " büyüğü " m_+ } olsun.

$$L_- Y_{lm_-} = 0$$

$$\vec{L}^2 Y_{lm_-} = \underbrace{\cancel{L_+ L_- Y_{lm_-}}}_{0} + L_z^2 Y_{lm_-} - \hbar L_z Y_{lm_-}$$

$$\Rightarrow l(l+1)\cancel{\hbar^2} = m_-^2 \cancel{\hbar^2} - \cancel{\hbar^2} m_-$$

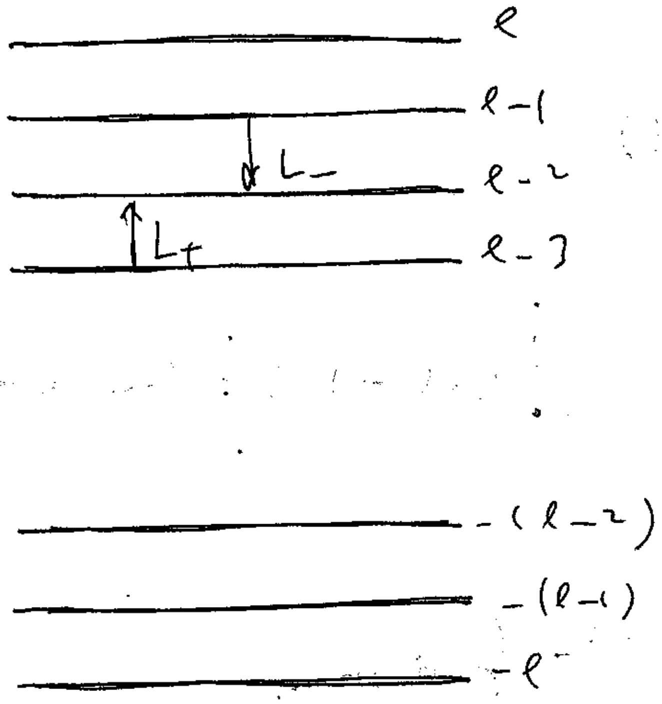
$$l(l+1) = m_-(m_- - 1)$$

$$m_-^2 - m_- - l(l+1) = 0 \Rightarrow m_- = \frac{1 \mp (2l+1)}{2}$$

(−) olur en küçük m_-

$$m_- = -\ell.$$

$$L_+ Y_{\ell m_+} = 0 \Rightarrow m_+ = +\ell$$ [illegible]



$$L_{\mp} Y_{\ell m}(\theta,\varphi) = C_{\pm}(\ell,m)\, Y_{\ell\, m\pm1}(\theta,\varphi)$$

$$\Rightarrow \langle L_\pm Y_{\ell m} \,|\, L_\pm Y_{\ell m}\rangle = |C_\pm|^2 \langle Y_{\ell m\pm1} \,|\, Y_{\ell m\pm1}\rangle$$

$$\Rightarrow \langle Y_{\ell m} \,|\, L_\mp L_\pm Y_{\ell m}\rangle = |C_\pm|^2$$

$$\langle Y_{\ell m} \,|\, L^2 - L_z^2 \mp \hbar L_z \,|\, Y_{\ell m}\rangle = |C_\pm|^2$$

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$$\left(\hbar^2 \ell(\ell+1) - \hbar^2 m^2 \mp m\hbar^2\right) = |C_\pm|^2$$

$$|C_\pm|^2 = \hbar^2 \left[\ell(\ell+1) - m(m \pm 1) \right]$$

$$C_\pm = \hbar \left[\ell(\ell+1) - m(m \pm 1) \right]^{1/2}$$

$$\boxed{L_\mp Y_{\ell m}(\theta,\varphi) = \hbar \left[\ell(\ell+1) - m(m \mp 1) \right]^{1/2} Y_{\ell m \mp 1}(\theta,\varphi)}$$

Küresel Harmonikler

$$Y_{\ell m}(\theta,\varphi) = \Theta_{\ell m}(\theta)\, e^{im\varphi}$$

$L_+ Y_{\ell m_+} = 0$ için. $\quad m_+ = \ell$

$$L_+ Y_{\ell\ell}(\theta,\varphi) = L_+\left[\Theta_{\ell\ell}(\theta)\, e^{i\ell\varphi} \right] = 0$$

$$\Rightarrow \quad \hbar e^{i\varphi} \left[\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\varphi} \right] \Theta_{\ell\ell}\, e^{i\ell\varphi} = 0$$

$$\hbar e^{i(\ell+1)\varphi} \left[\frac{d}{d\theta} + i^2 \ell \cot\theta \right] \Theta_{\ell\ell} = 0$$

BS K

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$$\frac{d\Theta_{\ell\ell}}{d\theta} = \ell \cot\theta \, \Theta_{\ell\ell}$$

$$\Rightarrow \quad \frac{d\Theta_{\ell\ell}}{\Theta_{\ell\ell}} = \ell \cot\theta \, d\theta = \ell \frac{\cos\theta}{\sin\theta} d\theta \qquad \sin\theta = u$$

$$\cos\theta \, d\theta = du$$

$$\ln \Theta_{\ell\ell} = \ell \ln \sin\theta$$

$$\boxed{\Theta_{\ell\ell} = (\sin\theta)^{\ell}}$$

$$Y_{\ell m} = C \, (L_-)^{\ell-m} \, Y_{\ell\ell}$$

$$= C \, (L_-)^{\ell-m} (\sin\theta)^{\ell} e^{i\ell\varphi}$$

Örneğin $\ell-1=m$ için.

$$Y_{\ell\,\ell-1} = C \underbrace{L_- (\sin\theta)^{\ell} e^{i\ell\varphi}}_{L_- Y_{\ell\ell}}$$

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$$L_- Y_{\ell\ell} = \hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)(\sin\theta)^\ell e^{i\ell\varphi}$$

$$= \hbar e^{i(\ell-1)\varphi}\left(-\frac{\partial}{\partial\theta} - \ell\cot\theta\right)(\sin\theta)^\ell$$

$$\left(\frac{d}{d\theta} + \ell\cot\theta\right)f(\theta) = \frac{1}{(\sin\theta)^\ell}\frac{d}{d\theta}\left[(\sin\theta)^\ell f(\theta)\right]$$

$$\Rightarrow \quad L_- Y_{\ell\ell} = \hbar\frac{e^{i(\ell-1)\varphi}}{(\sin\theta)^\ell}\left(\frac{-d}{d\theta}\right)\left[(\sin\theta)^\ell(\sin\theta)^\ell\right]$$

$$Y_{\ell\ell-1} = C'\frac{e^{i(\ell-1)\varphi}}{(\sin\theta)^\ell}\left(\frac{-d}{d\theta}\right)\left[(\sin\theta)^{2\ell}\right]$$

$$Y_{\ell,\ell-2} = C\,L_- Y_{\ell\ell-1}$$

$$= C''\hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)\frac{e^{i(\ell-1)\varphi}}{(\sin\theta)^\ell}\left(\frac{-d}{d\theta}\right)\left[(\sin\theta)^{2\ell}\right]$$

$$= C''' e^{i(\ell-2)\varphi}\underbrace{\left(\frac{\partial}{\partial\theta} + (\ell-1)\cot\theta\right)\frac{1}{(\sin\theta)^\ell}\left(\frac{-d}{d\theta}\right)\left[(\sin\theta)^{2\ell}\right]}_{\frac{1}{(\sin\theta)^{\ell-1}}\left(\frac{-d}{d\theta}\right)\left[(\sin\theta)^{\ell-1}\times\frac{1}{(\sin\theta)^\ell}\left(\frac{-d}{d\theta}\right)\left[(\sin\theta)^{2\ell}\right]\right]}$$

BS K

$$Y_{\ell\,\ell-2} = \bar{C}\,(-1)^2 \frac{e^{i(\ell-2)\varphi}}{(\sin\theta)^{\ell-1}} \frac{d}{d\theta}\left\{\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta)^{2\ell}\right\}$$

$$u = \cos\theta \quad \Rightarrow \quad \frac{d}{d\theta} = \frac{du}{d\theta}\frac{d}{du} = -\sin\theta\,\frac{d}{du}$$

$$\Rightarrow Y_{\ell\,\ell-1} = C'\,\frac{e^{i(\ell-1)\varphi}}{(\sin\theta)^{\ell-1}}\frac{d}{du}\left[(1-u^2)^{\ell}\right]$$

$$Y_{\ell\,\ell-2} = C''\,\frac{e^{i(\ell-2)\varphi}}{(\sin\theta)^{\ell-2}}\frac{d^2}{du^2}\left[(1-u^2)^{\ell}\right]$$

$$Y_{\ell m} = C\,\frac{e^{im\varphi}}{\underbrace{(\sin\theta)^m}_{(1-u^2)^{1/2}}}\left(\frac{d}{du}\right)^{\ell-m}\left[(1-u^2)^{\ell}\right]$$

$$\langle Y_{\ell m}|Y_{\ell m}\rangle = 1 = \int_0^{2\pi} d\varphi \int_{-1}^{+1} du\,|C|^2\left[\frac{1}{(1-u^2)^{m/2}}\right.$$

$$\left.\times\left(\frac{d}{du}\right)^{\ell-m}(1-u^2)^{\ell}\right]^2$$

$\ell-m$ defa kısmi integrasyon.

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$$Y_{\ell m}(\theta,\varphi) = (-1)^m \left[\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!} \right]^{1/2}$$

$$\times \; P_\ell^m(\theta)\, e^{im\varphi}$$

$$P_\ell^m(\theta) = (-1)^{m+\ell} \frac{(\ell+m)!}{(\ell-m)!} \frac{1}{2^\ell \ell!} \frac{1}{(1-u^2)^{m/2}} \frac{d^{\ell-m}}{du^{\ell-m}} (1-u^2)^\ell$$

$$Y_{\ell,-m} = (-1)^m Y_{\ell m}$$

$$P_\ell^{-m} = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m$$

$$|Y_{\ell\ell}|^2 = K^2 (\sin\theta)^{2\ell}$$

$m=\ell$: $\Rightarrow$ L_z en blijft minimale afwijking zodat $\vec{L}^2 \approx L_z^2$

klassiek alleen bij $\ell \gg 1$.

$$\frac{\vec{L}^2 - L_z^2}{\vec{L}^2} = \frac{1}{\ell} \to 0$$

✓

$\langle L_x^2 \rangle = \langle L_y^2 \rangle = 0$ minimale afwijk.

$x = r \sin\vartheta \cos\varphi$

$y = r \sin\vartheta \sin\varphi$

$z = r \cos\vartheta$

$$Y_{00} = 1/\sqrt{4\pi}$$

$$Y_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}}\, e^{\pm i\varphi} \sin\vartheta = \mp\sqrt{\frac{3}{8\pi}} \sin\vartheta\,(\cos\varphi \pm i \sin\varphi)$$

$$= \mp\sqrt{}\;\frac{x \pm iy}{r}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\vartheta = \sqrt{\frac{3}{4\pi}}\,\frac{z}{r}$$

BS K

AÇILIM TEOREMİ:

$Y_{\ell m}$ θ ve φ'nin ortonormal fonk.'larının bir tam kümesini oluşturur.

$$f(\theta,\varphi)=\sum\sum C_{\ell m}\,Y_{\ell m}(\theta,\varphi)$$

$$C_{\ell m}=\int d\Omega\, Y^*_{\ell m}(\theta,\varphi)\, f(\theta,\varphi)$$

Dahası f eğer bir açısal dalga fonk.'u ise

$$\int d\Omega\, |f|^2=1$$

olsun. O zaman $|C_{\ell m}|^2$: f ile tanımlanan durumda $\vec{L}^2$ ve L_z'nin eş zamanlı ölçümünün sırası ile $\ell(\ell+1)\hbar^2$ ve $m\hbar$ vermesinin olasılığıdır.

$\vec{L}^2$ ölçümünün $\ell(\ell+1)\hbar^2$'ye yol açmasının olasılığı

$$P(\ell)=\sum_{m=-\ell}^{+\ell}|C_{\ell m}|^2$$

$$\langle L_z\rangle=\sum_{\ell}\sum_{m} m\hbar\,|C_{\ell m}|^2$$

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abstract formda açılım teoremi

$$|\psi\rangle = \sum_{\ell m} c_{\ell m} |Y_{\ell m}\rangle$$

$$\langle Y_{\ell' m'} | Y_{\ell m}\rangle = \delta_{\ell\ell'}\,\delta_{mm'}$$

$$\Rightarrow \quad c_{\ell m} = \langle Y_{\ell m} | \psi\rangle$$

$$|\psi\rangle = \sum_{\ell}\sum_{m} |Y_{\ell m}\rangle\langle Y_{\ell m}|\psi\rangle$$

$$\Leftrightarrow \quad \sum_{\ell}\sum_{m} |Y_{\ell m}\rangle\langle Y_{\ell m}| = 1.$$

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Küresel Harmonikler cinsinden Düzlem Dalga :

$$\vec{\nabla}^2 \psi(\vec{r}) + k^2 \psi(\vec{r}) = 0$$

$$\psi(\vec{r}) = e^{i\vec{k}\vec{r}}$$

$$\psi(\vec{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} A_{\ell m}\, j_\ell(kr)\, Y_{\ell m}(\theta,\varphi)$$

$$\psi(\vec{r}) = e^{ikr\cos\theta}$$

bu m'ye bağlı değil yukarıda m=0 olur.

$$e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty} A_\ell\, j_\ell(kr)\, Y_{\ell 0}(\theta,\varphi)$$

$$Y_{\ell 0}(\theta,\varphi) = \left(\frac{2\ell+1}{4\pi}\right)^{1/2} P_\ell(\cos\theta)$$

$$e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty} A_\ell \left(\frac{2\ell+1}{4\pi}\right)^{1/2} j_\ell(kr)\, P_\ell(\cos\theta)$$

$$\int_{-1}^{+1} P_\ell P_{\ell'}\, dx = \frac{2\cancel{\ell}}{2\ell+1}\, \delta_{\ell\ell'}$$

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$$\int P_{l'}\, e^{ikr\cos\theta}\, dx = \sum_{l=} A_l \left(\frac{2l+1}{4\pi}\right)^{1/2} j_l(kr)$$

$$\frac{2}{2l+1}\,\delta_{ll'}$$

$$= \left(\frac{2}{2l'+1}\right) A_{l'} \left(\frac{2l'+1}{4\pi}\right)^{1/2} j_{l'}(kr)$$

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