BÖLÜM 17

REEL HİDROJEN ATOMU

Hidrojen benzeri atomların tartışmasında

$$H_0 = \frac{\vec{p}^2}{2\mu} - \frac{Ze^2}{r}$$

Hamiltonyenini almıştık. Proton hareketini ayrıntılı olarak, relativistik etkileri ve spin etkileri üzerinde yoğunlaşmadık.

$$K = \frac{\vec{p}_e^{\,2}}{2m_e} + \frac{\vec{p}_p^{\,2}}{2m_p} = \frac{1}{2}\vec{p}^{\,2}\left(\frac{1}{m_e} + \frac{1}{m_p}\right) = \frac{p^2}{2\mu}$$

$\vec{p} = \vec{p}_e = -\vec{p}_p$ alalım. (K.M. 'de)

$$\mu / m_e \simeq 1 - \frac{m_e}{m_p} \simeq 1 - 5.4 \times 10^{-4}$$

protonun elektrona göre daha ağır olduğu ve hareketsiz olduğunu söyleriz.

RELATİVİSTİK KİNETİK ENERJİ ETKİLERİ

Elektronun K.E.'si

$$K = \left[\vec{p}^{\,2}c^2 + m_e^2c^4\right]^{1/2} - m_e c^2$$

$$= m_e c^2 \left[1 + \frac{\vec{p}^{\,2}}{m_e^2 c^2}\right]^{1/2} - m_e c^2$$

BS K

$$K \simeq m_e c^2 \left[\cancel{1} + \frac{1}{2}\frac{\vec{p}^{\,2}}{m_e^2 c^2} - \frac{1}{8}\left(\frac{\vec{p}^{\,2}}{m_e^2 c^2}\right)^2 + \dots \right] - \cancel{m_e c^2}$$

$$\simeq \frac{\vec{p}^{\,2}}{2m_e} - \frac{1}{8}\frac{(\vec{p}^{\,2})^2}{m_e^3 c^2} + \dots$$

$$H_1 = -\frac{1}{8}\frac{(\vec{p}^{\,2})^2}{m_e^3 c^2}$$ pertürbasyon

indirgenmiş kütle etkisine kıyasla küçük.

Spin-yörünge etkileşmesi

Elektron hareketi açısından ile göreceği magnetik alan

$\vec{B} \sim \frac{\vec{V} \times \vec{E}}{c} \longrightarrow$ bu $\vec{S}$ ile etkileşir

$\downarrow$

$\vec{\mu}$

$$\boxed{\vec{\mu} = -\frac{eg}{2m_e c}\vec{S}}$$

BS K

$$-\vec{M}\cdot\vec{B} = \frac{eg}{2m_e c}\,\vec{S}\cdot\vec{B} = -\frac{e g^2}{2m_e c^2}\,\vec{S}\cdot\frac{\vec{V}\times\vec{E}}{c}$$

$$= +\frac{e}{m_e c^2}\,\vec{S}\cdot\vec{V}\times\vec{\nabla}\phi$$

$$= \frac{e}{(m_e c)^2}\,\vec{S}\cdot\underbrace{\vec{p}\times\vec{r}}_{-\vec{L}}\,\frac{1}{r}\frac{d\phi}{dr}$$

$$= -\frac{e}{m_e^2 c^2}\,\vec{S}\cdot\vec{L}\,\frac{1}{r}\frac{d\phi}{dr}$$

$$H_2 = -\frac{1}{2m_e^2 c^2}\,\vec{S}\cdot\vec{L}\,\frac{1}{r}\frac{d\left[e\phi(r)\right]}{dr}$$

Thomas çarpanı. (elektron durgun sisteme [illegible])

Şimdi Hidrojen benzeri atomların spektrumunu H_1 ve H_2'nin katkıları 1. mertebe pertürbasyon teorisi ile hesaplayalım.

$$H_1 = -\frac{1}{8}\frac{(\vec{p}^2)^2}{m_e^3 c^2} = -\frac{1}{2m_e c^2}\left(\frac{\vec{p}^2}{2m_e}\right)^2$$

$$= -\frac{1}{2m_e c^2}\left(H_0 + \frac{Ze^2}{r}\right)^2 = -\frac{1}{2m_e c^2}\left(H_0 + \frac{Ze^2}{r}\right)\left(H_0 + \frac{Ze^2}{r}\right)$$

BS K

$$\langle \phi_{nem} | H_1 | \phi_{nem} \rangle = -\frac{1}{2m_e c^2} \langle \phi_{nem} | (H_0 + \frac{Ze^2}{r})(H_0 + \frac{Ze^2}{r}) | \phi_{nem} \rangle$$

$$= -\frac{1}{2m_e c^2} \langle \phi_{nem} | \left[H_0^2 + (Ze^2)^2 \frac{1}{r^2} + Ze^2 \left(H_0 \frac{1}{r} + \frac{1}{r} H_0 \right) \right] | \phi_{nem} \rangle$$

$$= -\frac{1}{2m_e c^2} \left[E_n^2 + (Ze^2)^2 \left\langle \frac{1}{r^2} \right\rangle_{ne} + 2 E_n Ze^2 \left\langle \frac{1}{r} \right\rangle_{ne} \right]$$

$$= -\frac{1}{2m_e c^2} \left\{ \left[\frac{m_e c^2 (Z\alpha)^2}{2n^2} \right]^2 - 2 Ze^2 \frac{m_e c^2 (Z\alpha)^2}{2n^2} \frac{Z}{a_0 n^2} \right.$$

$$\left. + \frac{(Ze^2)^2}{1} \frac{Z^2}{a_0^3 n^3 (l+1/2)} \right\}$$

$$= -\frac{1}{2} m_e c^2 (Z\alpha)^2 \left\{ \frac{(Z\alpha)^2}{n^3 (l+1/2)} - \frac{3 (Z\alpha)^2}{4n^4} \right\}$$

$$\phi(r) = \frac{Ze}{r}$$

$$H_2 = -\frac{1}{2m_e^2 c^2} \vec{S} \cdot \vec{L} \frac{1}{r} \frac{e\, d\phi}{dr}$$

$$= + \frac{Z e^2}{2 m_e^2 c^2} \vec{S} \cdot \vec{L} \frac{1}{r^3}$$

$$\vec{J} = \vec{L} + \vec{S}$$

$$\vec{S} \cdot \vec{L} = \frac{1}{2} \left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2 \right)$$

BS K

$$\vec{S}\cdot\vec{L}\,\phi_{\substack{j=\ell+1/2\\ m_j=m+1/2}} = \frac{1}{2}\left(\vec{J}^2-\vec{L}^2-\vec{S}^2\right)\phi$$

$$= \frac{1}{2}\hbar^2\left[(\ell+\tfrac{1}{2})(\ell+\tfrac{3}{2})-\ell(\ell+1)-\tfrac{3}{4}\right]\phi$$

$$= \frac{\hbar^2}{2}\,\ell\,\phi$$

$$\vec{S}\cdot\vec{L}\,\phi_{\substack{j=\ell-1/2\\ m_j=m+1/2}} = \frac{\hbar^2}{2}\left[(\ell-\tfrac{1}{2})(\ell+1/2)-\ell(\ell+1)-\tfrac{3}{4}\right]\phi$$

$$= -\frac{\hbar^2}{2}(\ell+1)\,\phi$$

$$\langle\phi_{jm_j\ell}|H_2|\phi_{jm_j\ell}\rangle = \frac{Ze^2}{2m_e^2c^2}\frac{\hbar^2}{2}\begin{Bmatrix}\ell\\ -\ell-1\end{Bmatrix}$$

$$\times\underbrace{\int_0^\infty dr\,r^2 R_{n\ell}^2\frac{1}{r^3}}_{\dfrac{Z^3}{a_0^3}\dfrac{1}{n^3(\ell+1/2)(\ell+1)}\quad \ell\neq 0}$$

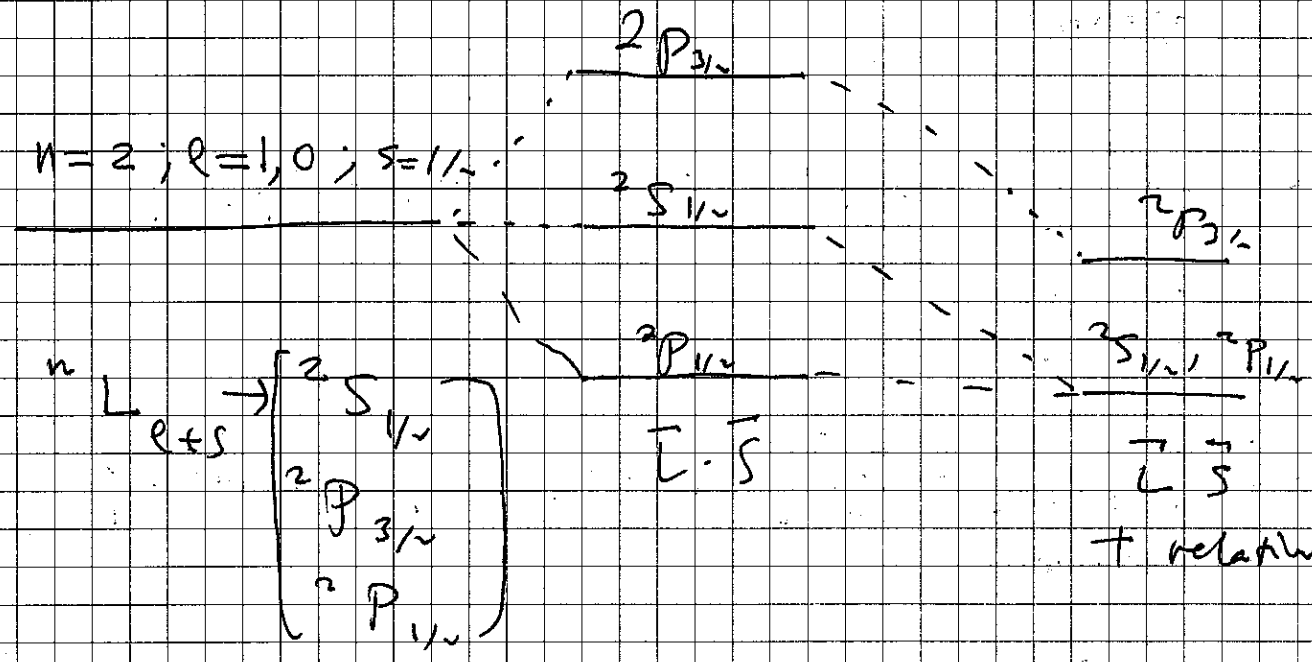
$$a_0 = \hbar/\mu c\alpha$$

$$\Delta E_2 = \frac{1}{4}m_e c^2(Z\alpha)^4\frac{1}{n^3\ell(\ell+1/2)(\ell+1)}\begin{Bmatrix}\ell\\ -\ell-1\end{Bmatrix}\quad \ell\neq 0$$

$$\int H_1 + H_2$$

$$\Delta E = -\frac{1}{2}m_e c^2(Z\alpha)^4\frac{1}{n^3}\left[\frac{1}{j+1/2}-\frac{3}{4n}\right]\quad \ell = j\pm 1/2$$

BS K $\ell=0 \Rightarrow$ Dirac Result. i.



Anomal Zeeman Etkisi

$$H_0 = \frac{\vec{p}^{\,2}}{2\mu} - \frac{Ze^2}{r} + \frac{1}{2m_e^2c^2}\,\frac{Ze^2}{r^3}\,\vec{L}\cdot\vec{S}$$

$$H_1 = \frac{e}{2m_e c}\,(\vec{L} + \underbrace{2}_{\text{gyromagnetic ratio}}\vec{S})\cdot\vec{B}$$

$$g = 2$$

$$\vec{\mu} = -\frac{eg}{2m_e c}\,\vec{S}$$

$$\vec{B} = B\hat{z}$$

$$\langle \phi_{j m_j l} \left| \frac{eB}{2m_e c}(L_z + 2S_z) \right| \phi_{j m_j l} \rangle$$

$$= \langle \; \left| \frac{eB}{2m_e c}(J_z + S_z) \right| \; \rangle = \frac{eB}{2m_e c}\left(\hbar m_j + \langle | S_z | \rangle\right)$$

BS K

$j = l + 1/2$

$$\left\langle \sqrt{\frac{l+m+1}{2l+1}}\, Y_{lm} \chi_+ + \sqrt{\frac{l-m}{2l+1}}\, Y_{lm+1} \chi_- \right| S_z \Big|$$

$$\sqrt{\frac{l+m+1}{2l+1}}\, Y_{lm} \chi_+ + \sqrt{\frac{l-m}{2l+1}}\, Y_{lm+1} \chi_- \Big\rangle$$

$$= \frac{\hbar}{2}\left(\frac{l+m+1}{2l+1} - \frac{l-m}{2l+1}\right) = \frac{\hbar}{2}\,\frac{2m+1}{2l+1} = \frac{\hbar m_j}{2l+1}$$

$j = l - 1/2$

$$\left\langle \sqrt{\frac{l-m}{2l+1}}\, Y_{lm} \chi_+ - \sqrt{\frac{l+m+1}{2l+1}}\, Y_{lm+1} \chi_- \right| S_z \Big|$$

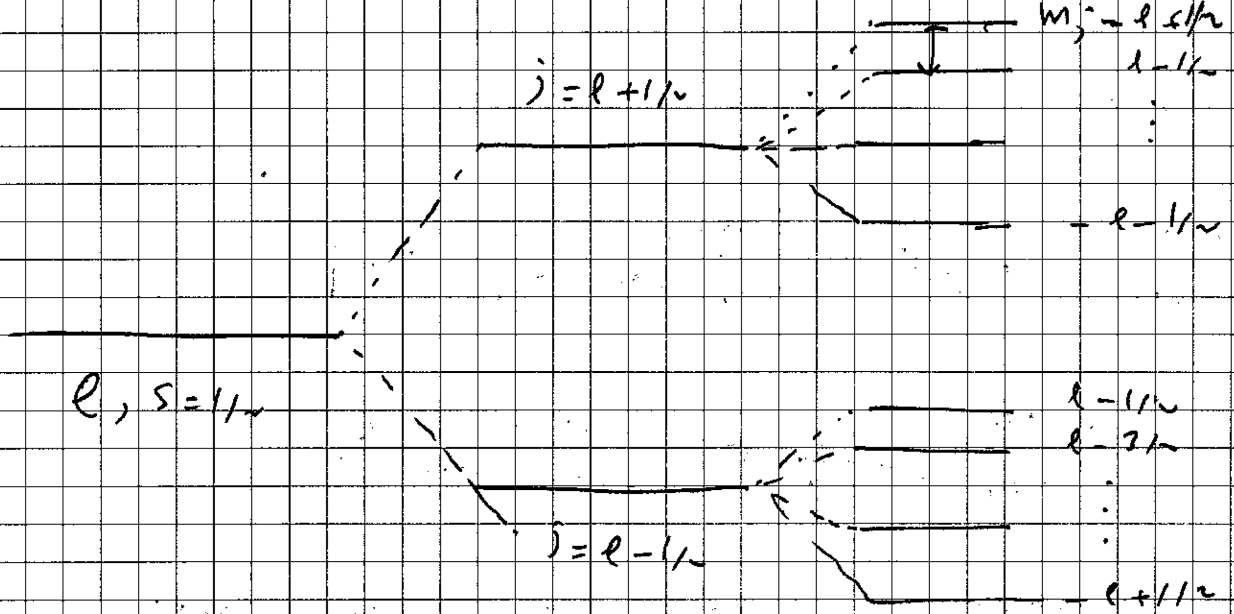
$$\sqrt{\frac{l-m}{2l+1}}\, Y_{lm} \chi_+ - \sqrt{\frac{l+m+1}{2l+1}}\, Y_{lm+1} \chi_- \Big\rangle$$

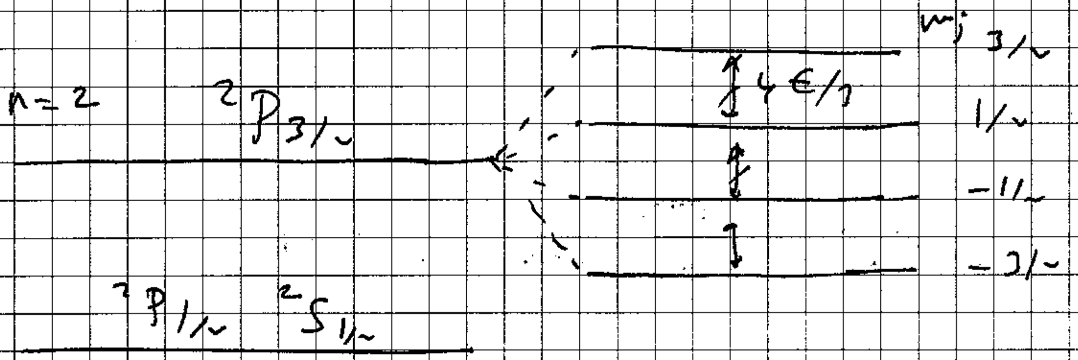
$$= \frac{\hbar^2}{2}\left(\frac{l-m}{2l+1} - \frac{l+m+1}{2l+1}\right) = -\frac{\hbar}{2}\,\frac{2m+1}{2l+1} = -\frac{\hbar m_j}{2l+1}$$

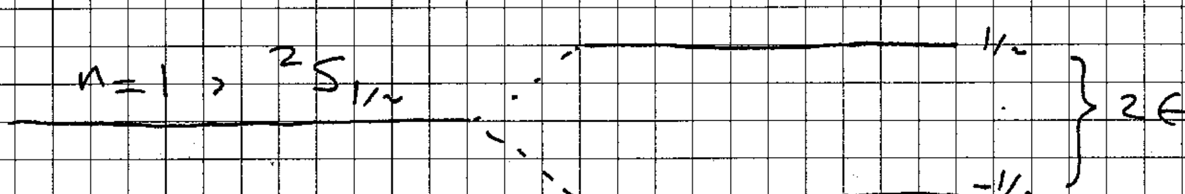
$$\Delta E = \underbrace{\frac{e\hbar B}{2m_e c}}_{\epsilon} m_j \left(1 \pm \frac{1}{2l+1}\right) \qquad j = l \pm 1/2$$

$$\Delta m_j = \pm 1, 0$$

BS K







Yarılma her çokulu için aynı değildir, Normal Zeeman olayı için elde ettiğimiz üç çizgiyi elde edemeyiz. Örneğin $n=2$ için $^2P_{3/2}$ 4 çizgiye ayrılır $^2P_{1/2}$ den iki kat daha sık. Dış alan kuvvetli ise ($\bar{L}\cdot\bar{S}$ ihmal edilebilir) basitçe spinlerle çarpılmış hydrojenik dalga fonksiyonları kullanılabilir.

BS K

L_z ve S_z'nin özdeğerlerini m_ℓ ve m_s olarak adlandırırsak H_1'in beklenen değeri $(\vec{B}=B\hat{z})$

$$\langle H_1 \rangle = \frac{e\hbar B}{2mc}\,(m_\ell + 2m_s)$$

Böylece $n=2$, $\ell=1$; $m_\ell=\pm1,0$ ve $m_s=\pm 1/2$ değerlerine karşı gelen 5 çizgiye ayrılır.

AŞIRI İNCE YAPI

○ $\vec{L}$, $\vec{S}$ kuplajının sonucu olduğu ince yapıya ek olarak çekirdeğin magnetik dipol mom.'i tarafından üretilen magnetik alandan dolayı daimi bir Zeeman yarılması olan çok ince bir aşırı ince yarılma da vardır. $\vec{I}$ çekirdeğin spin olsun magnetik dipol mom. op.'ü

$$\vec{M} = \frac{Zeg_N}{2M_p c}\,\vec{I}$$

○ $$\vec{A}(\vec{r}) = -\frac{1}{4\pi}\,(\vec{M}\times\vec{\nabla})\,\frac{1}{r}$$ nokta dipol'ün

$$\vec{B}=\vec{\nabla}\times\vec{A} = -\frac{\vec{M}}{4\pi}\,\vec{\nabla}^2\frac{1}{r} + \frac{1}{4\pi}\,\vec{\nabla}(\vec{M}\cdot\vec{\nabla})\,\frac{1}{r}$$

$$\Rightarrow B_i = -M_i\frac{1}{4\pi}\nabla^2\frac{1}{r} + \frac{1}{4\pi}M_j\frac{\partial^2}{\partial x_i\,\partial x_j}\frac{1}{r}$$

$$= M_i\,\delta(\vec{r}) + \frac{1}{4\pi}\,M_j\,\partial^2_{x_j x_i}\,\frac{1}{r}$$

BS K

$$H_{hf} = -\vec{\mu}_e \cdot \vec{B} = -\mu_{e_i} \mu_i \delta(r) + \frac{1}{4\pi} \mu_{e_i} \mu_j \partial^2_{x_i x_j} \frac{1}{r}$$

Basit bir boyut analizi pertürbasyonun $\vec{\mu}_e \cdot \vec{\mu} / a_0^3$ şeklinde olacağını gösterir

$$\langle H_{hf} \rangle \simeq \frac{Ze^2 g_N}{m_e M_N c^2} \hbar^2 \left(\frac{Z\alpha m_e c}{\hbar} \right)^3 \approx g_N (Z\alpha)^4 m_e c^2 \underbrace{\left(\frac{m_e}{m_N}\right)}_{}$$

$\lfloor -5$ yarılma civarı küçük

Hydrojen atomunun $l=0$ durumunun yarılmasını inceleyelim.

$$\int d^3r \, |\phi_{n0}(\vec{r})|^2 \left(-\mu_{e_i} \mu_i \, \delta(\vec{r}) + \frac{1}{4\pi} \mu_{e_i} \mu_j \, \partial_{x_i x_j} 1/r \right)$$

$|\phi_{n0}(r)|^2$ nin küresel simetrisinden,

$$\int d^3r \, |\phi_{n0}(r)|^2 \partial^2_{x_i x_j} \frac{1}{r} = \frac{1}{3} \delta_{ij} \int d^3r \, |\phi_{n0}|^2 \underbrace{\nabla^2 \frac{1}{r}}_{-4\pi\delta(r)}$$

$$= \cancel{-\frac{4\pi}{3}\delta}$$

$$= -\frac{4\pi}{3} \delta_{ij} \int d^3r \, |\phi_{n0}(r)|^2 \, \delta(r) \qquad -1 - \frac{1}{3}$$

$$= -\frac{4\pi}{3} |\phi_{n0}(0)|^2$$

$$= -\overset{?}{\frac{4}{3}} \mu_{e_i} \mu_i \, |\phi_{n0}(0)|^2$$

BS K

$$\langle \phi_{n0} | H_{hf} | \phi_{n0} \rangle = -\frac{4}{3} \vec{M}_e \cdot \vec{M} \, \underbrace{|\phi_{n0}(0)|^2}_{\frac{4}{n^3}\left(\frac{Z\alpha m_e c}{\hbar}\right)^2}$$

$$\langle H \rangle = \frac{4}{3} g_N \frac{m_e}{m_N} (Z\alpha)^4 m_e c^2 \frac{1}{n^3} \frac{\vec{S}\cdot\vec{I}}{\hbar^2}$$

$$\vec{F} = \vec{S} + \vec{I}$$

$$\frac{\vec{S}\cdot\vec{I}}{\hbar^2} = \frac{\vec{F}^2 - S^2 - I^2}{2\hbar^2} = \frac{F(F+1) - 3/4 - I(I+1)}{2}$$

$$= \frac{1}{2} \begin{cases} I & F = I + 1/2 \\ -I-1 & F = I - 1/2 \end{cases}$$

Hyd: $g_N = g_p \simeq 5.56$

BS K

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BS K

prb. 17.1

$$H_{so} = \frac{1}{2m^2c^2}\, \vec{S}\cdot\vec{L}\, \frac{1}{r}\, \frac{dV(r)}{dr}$$

küresel osilatörün spektrumu üzerine etkisi?

$$H = -\frac{\hbar^2}{2m}\vec{\nabla}^2 + \frac{1}{2}kr^2 \qquad k = m\omega^2$$

$$= -\frac{\hbar^2}{2m}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\right\} + \frac{1}{2}kr^2$$

$$H\psi = E\psi \qquad \psi = R(r)\,Y(\theta,\varphi)$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{mk}{\hbar^2}r^4 + \frac{2m}{\hbar^2}Er^2$$

$$= -\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right] = \ell(\ell+1)$$

$$Y = \Theta\,\Phi$$

$$\frac{1}{\Theta}\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\sin^2\theta$$

$$= -\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = m^2$$

$$Y = Y_\ell^m(\theta,\varphi) = (-1)^m\sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\theta)\,e^{im\varphi}$$

BS K

(2)

$$r = \sqrt{\hbar/m\omega}\ \rho \qquad 2E/\hbar\omega = \lambda$$

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) + \left(\lambda - \rho^2 - \frac{\ell(\ell+1)}{\rho^2}\right)R = 0$$

$$R = e^{-\rho^2/2}\,\Phi(\rho)$$

$$\frac{d^2\Phi}{d\rho^2} + 2\left(\frac{1}{\rho} - \rho\right)\frac{d\Phi}{d\rho} + \left(-3 + \lambda - \frac{\ell(\ell+1)}{\rho^2}\right)\Phi = 0$$

$$\Phi(\rho) = \sum_{k=0}^{\infty} a_k \rho^{k+s}$$

$$s = \begin{cases} \ell \\ -(\ell+1) \end{cases}$$

$\rho = 0$ da Φ sürekli $-(\ell+1)$ çözüm olamaz

$$\lambda - 3 = 2n \Rightarrow \lambda = 2n+3 = \frac{2E}{\hbar\omega}$$

$$E_n = \left(n + \frac{3}{2}\right)\hbar\omega$$

$$\Phi(\rho) = \rho^{\ell} F(\rho) \qquad z = \rho^2$$

$$z\frac{d^2F}{dz^2} + \left(\ell + \frac{3}{2} - z\right)\frac{dF}{dz} + \overbrace{\frac{n-\ell}{2}}^{k} F = 0$$

$$F = L_k^{\ell+\frac{1}{2}}(z) = L_{\frac{n-\ell}{2}}^{\ell+\frac{1}{2}}(\rho^2)$$

BS K

$$R(r) = N_{nl}\, e^{-m\omega r^2/2\hbar} \left[\sqrt{\frac{m\omega}{\hbar}}\, r\right]^{l} L^{l+\frac{1}{2}}_{\frac{n-l}{2}}\left(\frac{m\omega}{\hbar} r^2\right)$$

$$= \sqrt{2}\left(\frac{m\omega}{\hbar}\right)^{3/4}\left[\Gamma\left(l+\frac{2}{3}\right)\begin{pmatrix}\frac{n+l+1}{2}\\ \frac{n-l}{2}\end{pmatrix}\right]^{1/2}$$

$$\frac{dV(r)}{dr} = kr$$

$$H_{SO} = \frac{k}{2m^2c^2}\,\vec{S}\cdot\vec{L}$$

$$\vec{S}\cdot\vec{L} = \frac{1}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)$$

$$\vec{S}\cdot\vec{L}\,\psi_{\substack{j=l+1/2\\ m_j=m+1/2}} = \frac{\hbar^2}{2}\, l\, \psi_{\substack{j=l+1/2\\ m_j=m+1/2}}$$

$$\vec{S}\cdot\vec{L}\,\psi_{j=l-1/2} = -\frac{\hbar^2}{2}(l+1)\,\psi_{j=l-1/2}$$

$$\langle H_{SO}\rangle = \frac{k}{2m^2c^2}\,\frac{\hbar^2}{2}\begin{cases} l \\ -(l+1)\end{cases}$$

BS K

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Extra: nötron ve protondan oluşan bir sist.

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$\vec{S}_1 = \frac{\hbar}{2}\vec{\sigma}_1 \qquad \vec{S}_2 = \frac{\hbar}{2}\vec{\sigma}_2$$

olmak üzere

$$V(r) = V_1(r) + V_2(r)\left(3\frac{(\vec{\sigma}_1\cdot\vec{r})(\vec{\sigma}_2\cdot\vec{r})}{r^2} - \vec{\sigma}_1\cdot\vec{\sigma}_2\right) + V_3\,\vec{\sigma}_1\cdot\vec{\sigma}_2$$

$$\vec{G}\,? = \frac{2}{\hbar^2}\left[\frac{3(\vec{S}\cdot\vec{r})^2}{r^2} - \vec{S}^2\right]$$

dipol-dipol.

$$\vec{S} = \vec{S}_1 + \vec{S}_2$$

$$\Rightarrow (\vec{S}\cdot\vec{r})^2 = \frac{\hbar^2}{4}(\vec{\sigma}_1\cdot\vec{r} + \vec{\sigma}_2\cdot\vec{r})^2$$

$$= \frac{\hbar^2}{4}\left[(\vec{\sigma}_1\cdot\vec{r})^2 + (\vec{\sigma}_2\cdot\vec{r})^2 + 2(\vec{\sigma}_1\cdot\vec{r})(\vec{\sigma}_2\cdot\vec{r})\right]$$

$$(\vec{\sigma}_1\cdot\vec{r})^2 = \left[x\sigma_{1x} + y\sigma_{1y} + z\sigma_{1z}\right]^2$$

$$= x^2 + y^2 + z^2 = r^2$$

$$(\vec{S}\cdot\vec{r})^2 = \frac{\hbar^2}{2}\left(\vec{r}^2 + (\vec{\sigma}_1\cdot\vec{r})(\vec{\sigma}_2\cdot\vec{r})\right)$$

$$(\vec{\sigma}_1\cdot\vec{r})(\vec{\sigma}_2\cdot\vec{r}) = \frac{2}{\hbar^2}(\vec{S}\cdot\vec{r})^2 - r^2$$

$$\frac{(\vec{\sigma}_1\cdot\vec{r})(\vec{\sigma}_2\cdot\vec{r})}{r^2} = \frac{2}{\hbar^2}\frac{(\vec{S}\cdot\vec{r})^2}{r^2} - 1$$

BS K

$$\bar{S}^2 = \frac{\hbar^2}{4}\left(\bar{\sigma}_1^2 + \bar{\sigma}_2^2 + 2\bar{\sigma}_1\cdot\bar{\sigma}_2\right) = \frac{\hbar^2}{4}\left(6I + 2\bar{\sigma}_1\cdot\bar{\sigma}_2\right)$$

$$= \frac{\hbar^2}{2}\left(3I + \bar{\sigma}_1\cdot\bar{\sigma}_2\right)$$

$$\bar{\sigma}_1\cdot\bar{\sigma}_2 = \frac{2}{\hbar^2}\bar{S}^2 - 3I$$

$$\frac{3(\bar{\sigma}_1\cdot\bar{r})(\bar{\sigma}_2\cdot\bar{r})}{r^2} = \frac{6}{\hbar^2}\frac{(\bar{S}\cdot\bar{r})^2}{r^2} - 3I$$

$$\bar{\sigma}_1\cdot\bar{\sigma}_2 = \frac{2}{\hbar^2}\bar{S}^2 - 3I$$

$-$

$$\cdots = \frac{6}{\hbar^2}\frac{(\bar{S}\cdot\bar{r})^2}{r^2} - \frac{2}{\hbar^2}\bar{S}^2 = \frac{2}{\hbar^2}\left[\frac{3(\bar{S}\cdot\bar{r})^2}{r^2} - \bar{S}^2\right]$$

BS K

BS K