Olasılık fonksiyonu (genliği)

Parçacıklar için dalga denkleminin çözümleri olan dalga fonksiyonları olasılık genliği olarak yorumlanır.

(1B) $\dfrac{\partial^2 \psi_x}{\partial t^2} = v^2 \dfrac{\partial^2 \psi_x}{\partial z^2}$ $\qquad \psi_x(z,t) = A\, e^{i(kz - \omega t + \delta)}$

Klasik dalga denklemi $\qquad \omega = kv$

3B $\boxed{\dfrac{\partial^2 \psi}{\partial t^2} = v^2 \nabla^2 \psi}$ $\qquad \psi(\vec{r},t) = A\, e^{i(\vec{k}\cdot\vec{r} - \omega t + \delta)}$

$\vec{r} = (x, y, z)$

$\psi(\vec{r},t)$: Parçacığın t anında $\vec{r}$ civarında bulunma olasılık genliği

$|\psi(\vec{r},t)|^2 = |\psi^* \psi|$: Olasılık yoğunluğu

$P_V = \int_V |\psi(\vec{r},t)|^2 d^3r$ = Parçacığın V hacmi içinde bulunma olasılığı (t anında)

$P = \int_{\text{Tüm uzay}} |\psi(\vec{r},t)|^2 d^3r = 1 = \dfrac{100}{100}$ (Normalizasyon) Boylandırma

EK:

Gaussian olasılık yoğunluğu

$$|\psi(x)|^2 = \frac{1}{\sqrt{2\pi}} e^{-(x-3)^2 \frac{1}{2}}$$

ortalama konum $\langle x \rangle = \int_{-\infty}^{\infty} x |\psi|^2 dx$

$x - 3 = y$

$$\langle x \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (y+3) e^{-y^2 \frac{1}{2}} dy = 0 + \frac{3}{\sqrt{2\pi}} 2\int_0^{\infty} e^{-y^2/2} dy$$

— o —

$$= 2\int_0^{\infty} e^{-\alpha z^2} dz = \sqrt{\frac{\pi}{\alpha}} \qquad 4\int_0^{\infty} dy_1 \int_0^{\infty} dy_2\, e^{-\alpha (y_1^2 + y_2^2)}$$

$$I^2 = 4\int_0^{\pi/2} d\theta \int_0^{\infty} \frac{dr^2}{2} e^{-\alpha r^2} \qquad y_1^2 + y_2^2 = r^2 \qquad dy_1\, dy_2 = r\,dr\,d\theta$$

$$I^2 = 4\int_0^{\pi/2} d\theta \left(\frac{1}{2} \left(-\frac{1}{\alpha}\right) e^{-\alpha r^2} \right]_0^{\infty} = \frac{\pi}{\alpha} \qquad I = \sqrt{\frac{\pi}{\alpha}}$$

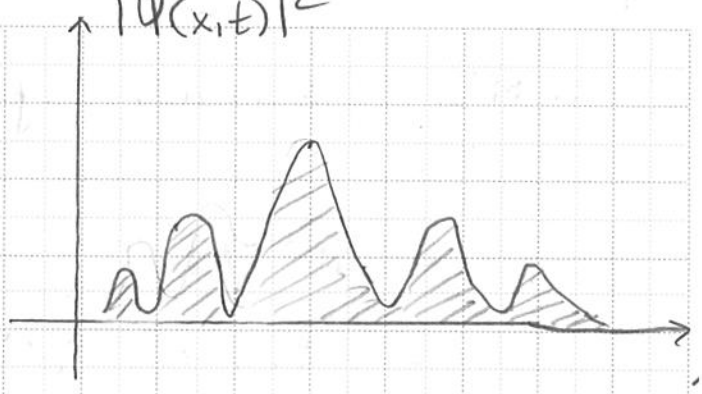
— o —

$$\langle x \rangle = \frac{3}{\sqrt{2\pi}} \sqrt{2\pi} = 3$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx = 10$$

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{10 - 9} = 1 = \Delta x_s$$

Örnek: Grafiğe göre parçacık sadece taralı bölge içinde bir yerde bulunma olasılığına sahiptir.



Ortalama Konum

$$\langle x \rangle = \bar{x} = \int x \, |\Psi(x,t)|^2 \, dx$$

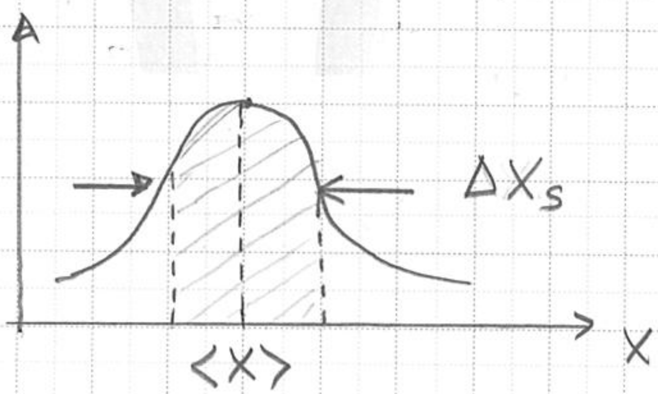
Ortalama değerden Sapma

$$\Delta x = (x - \langle x \rangle)$$

Standart Sapma $\Delta x_s = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}$

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

Örnek:



Girişim

$$P = |\Psi_1 + \Psi_2|^2 = |\Psi_1|^2 + |\Psi_2|^2 + \underbrace{\Psi_1^* \Psi_2 + \Psi_1 \Psi_2^*}_{\text{girişim terimi}}$$

(Superposition)

Fourier Dönüşümü

$$f(x) = \frac{1}{\sqrt{2\pi}} \int e^{+ikx} g(k)\, dk$$

$$g(k) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} f(x)\, dx$$

Dalga, $k = \frac{2\pi}{\lambda}$

$$f(x) = \frac{1}{\sqrt{2\pi\hbar}} \int e^{\frac{ipx}{\hbar}} g(p)\, dk$$

$$g(p) = \frac{1}{\sqrt{2\pi\hbar}} \int e^{-\frac{ipx}{\hbar}} f(x)\, dx$$

parçacık, $p = \hbar k$

Dalga- Parçacık İlişkisi

	Parçacık	Dalga	Dağınım Bağıntısı	
	E, p	ω, k ν, λ		
Işık	$E = h\nu$ $p = \hbar k$	ν, λ	$\omega = kc$ $E = pc$	
Madde	E, p	$\lambda = \frac{h}{p}$ $\omega = \frac{E}{\hbar}$	$E = p^2/2m$ $\omega = \frac{\hbar k^2}{2m}$	

Momentum olasılık genliği

$\varphi(p)$: parçacığın p momentumuna sahip olma olasılık genliği

$|\varphi(p)|^2$: p momentumuna sahip olma olasılık yoğunluğu

$\int_{\Delta p} |\varphi(p)|^2 dp$: Δp aralığında p momentumuna sahip olma olasılığı

EK:

$$\varphi(p)=\frac{1}{\sqrt{2\pi}}\int dx\, e^{-\frac{x^2}{4\alpha}}\left(\frac{1}{2\pi\alpha}\right)^{1/4} e^{-ipx}$$

$$=\frac{1}{\sqrt{2\pi}}\left(\frac{1}{2\pi\alpha}\right)^{1/4}\int_{-\infty}^{\infty} dx\, e^{-\left[ipx+\frac{x^2}{4\alpha}\right]}$$

$$(cx+d\,ip)^2=c^2x^2-d^2p^2+(2cd)ipx$$

$$c=\frac{1}{2\sqrt{\alpha}} \qquad 2cd=1 \qquad d=\frac{1}{2c}=\sqrt{\alpha}$$

$$cx+dip=x'$$

$$\varphi(p)=\left[\;''\;\right] e^{-\alpha p^2}\int_{-\infty}^{\infty}\frac{(c\,dx)}{c}\, e^{-x'^2}$$

$$=\left(\frac{1}{(2\pi)^2\,2\pi\alpha}\right)^{\frac{1}{4}} 2\sqrt{\alpha}\left(\int_{-\infty}^{\infty} e^{-x'^2}\,dx'\right) e^{-\alpha p^2}$$

$$=\left(\frac{2^4\alpha^2}{2^2\pi^2\,2\pi\alpha}\right)^{\frac{1}{4}}\sqrt{\pi}\; e^{-\alpha p^2}$$

$$\boxed{\varphi(p)=\left(\frac{2\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\alpha p^2}}$$

$\varphi(p)$ Nasıl Bulunur?

Eğer $\psi(x)$ biliniyorsa Fourier dönüşümü ile $\varphi(p)$ bulunur.

$$\varphi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \, \psi(x) \, dx \qquad (\hbar = sb)$$

$$\psi(x) = \frac{}{\sqrt{2\pi\hbar}} \int e^{ipx/\hbar} \, \varphi(p) \, dp$$

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Örnek:

$$\psi(x) = \left(\frac{1}{2\pi\alpha}\right)^{1/4} e^{-\frac{x^2}{4\alpha}} \qquad \hbar = 1$$

$$\varphi(p) = \left(\frac{2\alpha}{\pi}\right)^{1/4} e^{-\alpha p^2}$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x \, |\psi(x)|^2 \, dx = 0$$

$$\Delta x^2 = \langle x^2 \rangle - \langle x \rangle^2 = \langle x^2 \rangle = \int x^2 \, |\psi(x)|^2 dx = \alpha$$

$$\langle p \rangle = 0 \qquad \Delta p^2 = \frac{1}{4\alpha}$$

EK:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = 2\int_{0}^{\infty} e^{-x^2} dx^2$$

$$\left(2\int_{0}^{\infty} e^{-x^2} dx\right)^2 = 4\int_{0}^{\infty} dx\,dy\, e^{-(x^2+y^2)} = \int_{0}^{\infty} r\,dr\, e^{-r^2}\int_{0}^{2\pi} d\phi$$

$$x^2+y^2=r^2 \qquad dx\,dy = r\,dr\,d\phi$$

$$\int_{0}^{\infty} r\,dr\, e^{-r^2}\int_{0}^{2\pi} d\phi = \frac{1}{2}(2\pi) = \pi$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\boxed{\Delta x \, \Delta p = \frac{1}{2}}$$

Δx : Konumdaki belirsizlik

Δp : Momentumdaki belirsizlik

Belirsizlik ilkesi

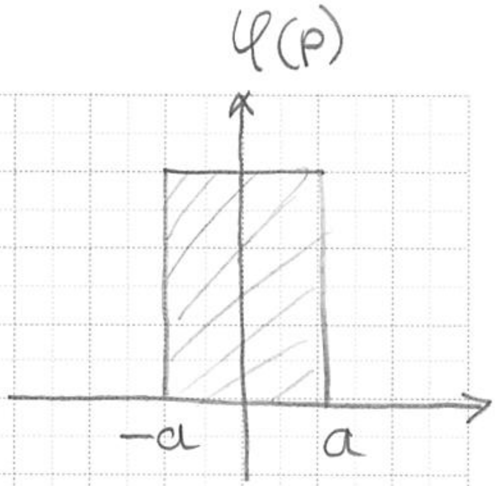
Kuantum fiziğinin koruyucusu

$$\boxed{\Delta x \, \Delta p \geqslant \frac{\hbar}{2}}$$

Heisenberg (1927) Belirsizlik ilkesi

Örnek:

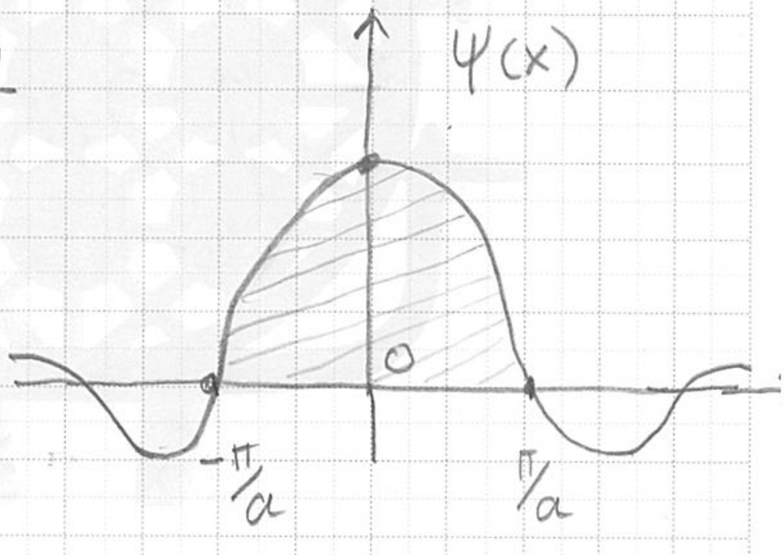
$$\varphi(p) = \begin{cases} \frac{1}{\sqrt{2a}} & p < |a| \\ 0 & p > |a| \end{cases}$$



$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int e^{ipx} \varphi(p)\, dp$$

$$= \frac{1}{\sqrt{4\pi a}} \frac{e^{ipx}}{ix} \Big]_{-a}^{a} = \sqrt{\frac{a}{\pi}} \frac{\sin(ax)}{ax}$$

$$\psi(x) = \sqrt{\frac{a}{\pi}} \frac{\sin(ax)}{ax}$$



$\Delta p = 2a$

$\Delta x = \frac{2\pi}{a}$

$\Delta x \, \Delta p = 4\pi > 1$

Dirac Delta Fonksiyonu

$$\left.\begin{aligned}
\psi(x) &= \frac{1}{\sqrt{2\pi\hbar}} \int \varphi(p)\, e^{ipx/\hbar}\, dp \\
\varphi(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int \psi(x)\, e^{-ipx/\hbar}\, dx
\end{aligned}\right\} \text{Fourier Dönüşümü}$$

İkinci ifadeyi birincide yerine koyalım

$$\psi(x) = \frac{1}{2\pi\hbar} \int e^{ipx/\hbar} \int dp \int dx'\, \psi(x')\, e^{-ipx'/\hbar}$$

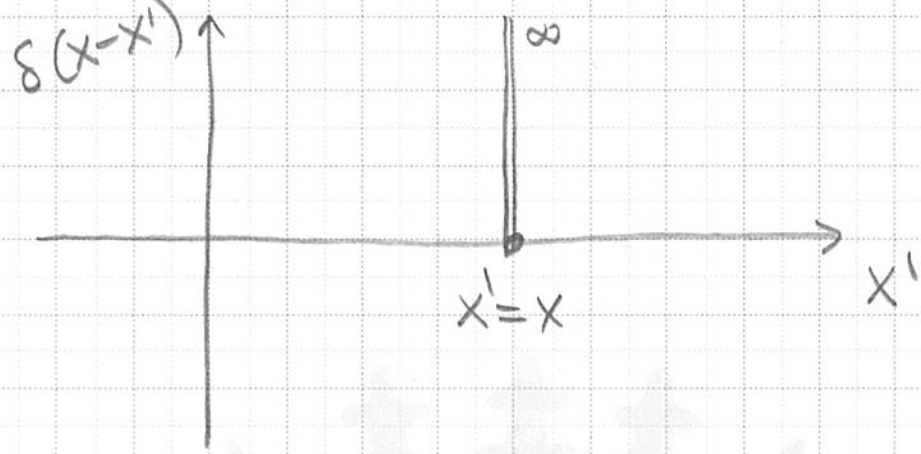
$$= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dx'\, \psi(x') \left[\int_{-\infty}^{\infty} dp\, e^{ip(x-x')/\hbar} \right]$$

$$= \int_{-\infty}^{\infty} dx'\, \psi(x')\, \delta(x-x') = \psi(x)$$

$$\delta(x-x') = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp\, e^{ip(x-x')/\hbar}$$

Dirac Delta fonk.

$$\delta(x-x') = \begin{cases} x' = x & \rightarrow \quad \infty \\ x' \neq x & \quad 0 \end{cases}$$



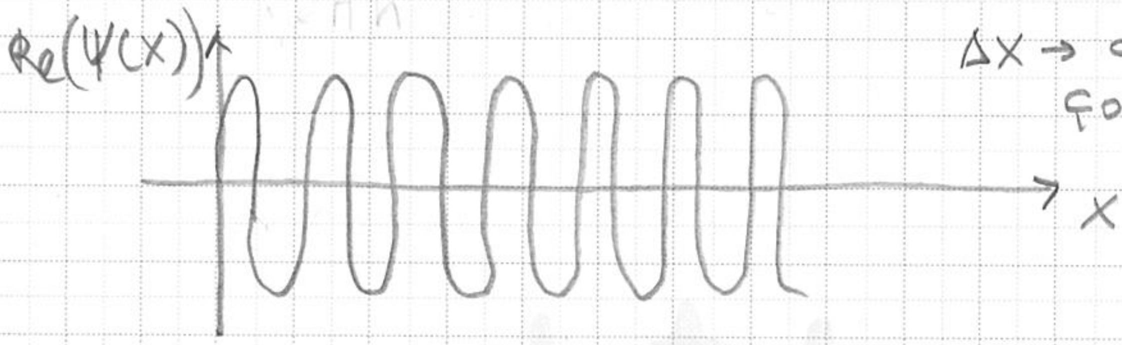
$$f(x) = \int_{-\infty}^{\infty} \delta(x-x')\, f(x')\, dx'$$

$$\varphi(p) = \int_{-\infty}^{\infty} \delta(p'-p)\, \varphi(p')\, dp'$$

$$\int_{-\infty}^{\infty} \delta(x-x')\, dx' = 1 \quad , \quad \int_{-\infty}^{\infty} \delta(x')\, dx' = 1$$

Düzlem Dalganın Fourier Dönüşümü

$$\psi(x) = A\, e^{ip_0 x/\hbar} = A\, e^{ik_0 x}$$



$\Delta x \to \infty$

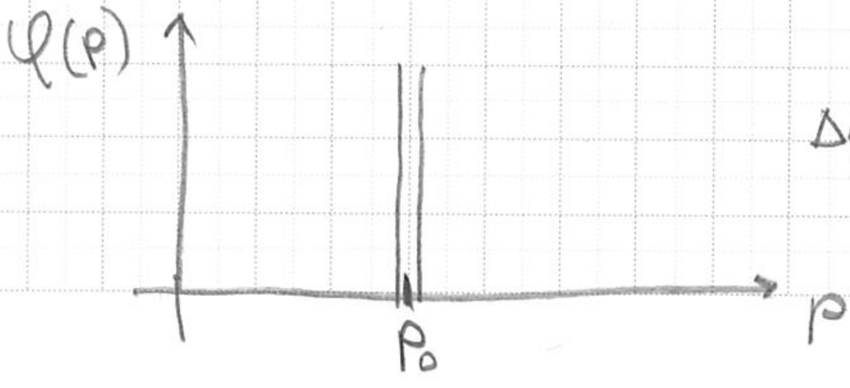
çok belirsi

$$\varphi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int dx\, e^{-ipx/\hbar} \left(A\, e^{ip_0 x/\hbar}\right)$$

$$= \frac{A}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx\; e^{-i(p-p_0)x/\hbar}$$

$$\varphi(p) = \sqrt{2\pi\hbar}\, A \quad \delta(p-p_0)$$

Eğer $A = \frac{1}{\sqrt{2\pi\hbar}}$ ise $\varphi(p) = \delta(p-p_0)$ ol



$\Delta p \to 0$ çok belirl

Konum ve Momentum Uzayında Olasılık fonksiyonlarının Bazı özellikleri:

$$\int |\psi(x)|^2 dx = \int |\varphi(p)|^2 dp$$

Parseval Teoremi

İspat:

$$\int dx\, \psi^*(x) \left(\int \frac{1}{\sqrt{2\pi}} e^{ipx} \varphi(p)\, dp \right)$$

$$= \int dp\, \varphi(p) \left(\int \frac{1}{\sqrt{2\pi}} e^{ipx} \psi^*(x)\, dx \right)$$

$$= \int dp\, \varphi(p)\, \varphi^*(p) = \int dp\, |\varphi(p)|^2$$

—— o ——

Olasılık Fonksiyonlarının türevleri

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int e^{ipx/\hbar} \varphi(p)\, dp$$

$$\frac{d\psi(x)}{dx} = \frac{1}{\sqrt{2\pi\hbar}} \int dp\, e^{ipx/\hbar} \left(\frac{ip}{\hbar} \varphi(p)\right)$$

Yani

$\frac{d\psi}{dx}$ 'in Fourier dönüşümü $\rightarrow \frac{ip}{\hbar}\varphi(p)$

$$\frac{d\psi}{dx} \xrightarrow{FD} \frac{ip}{\hbar}\varphi(p)$$

—— o ——

$$\frac{d^2\psi}{dx^2} = \frac{1}{\sqrt{2\pi\hbar}} \int dp\, e^{ipx/\hbar} \left(-\frac{p^2}{\hbar^2} \varphi(p)\right)$$

$$\frac{d^2\psi}{dx^2} \xrightarrow{FD} \left[-\frac{p^2}{\hbar^2} \varphi(p)\right]$$

—— o ——

$$\varphi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int dx\, e^{-ipx/\hbar}\, \psi(x)$$

$$\frac{d\varphi(p)}{dp} = \frac{1}{\sqrt{2\pi\hbar}} \int dx\, e^{-ipx/\hbar} \left(\frac{-ix}{\hbar}\psi(x)\right)$$

$$\frac{d\varphi(p)}{dp} \xrightarrow{FD} \left[\frac{-ix}{\hbar}\psi(x)\right]$$

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$$\frac{d^2\varphi(p)}{dp^2} = \frac{1}{\sqrt{2\pi\hbar}} \int dx\, e^{\frac{-ipx}{\hbar}} \left(-\frac{x^2}{\hbar}\psi(x)\right)$$

$$\frac{d^2\varphi(p)}{dp^2} \xrightarrow{FD} \left[-\frac{x^2}{\hbar}\psi(x)\right]$$

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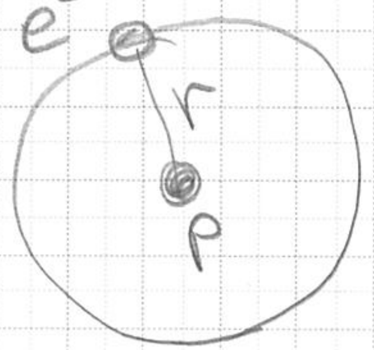
Newton Atomu (Klasik Fizik)



$$-\frac{e^2}{r^2}\hat{r} = m\omega^2 r\,(-\hat{r})$$

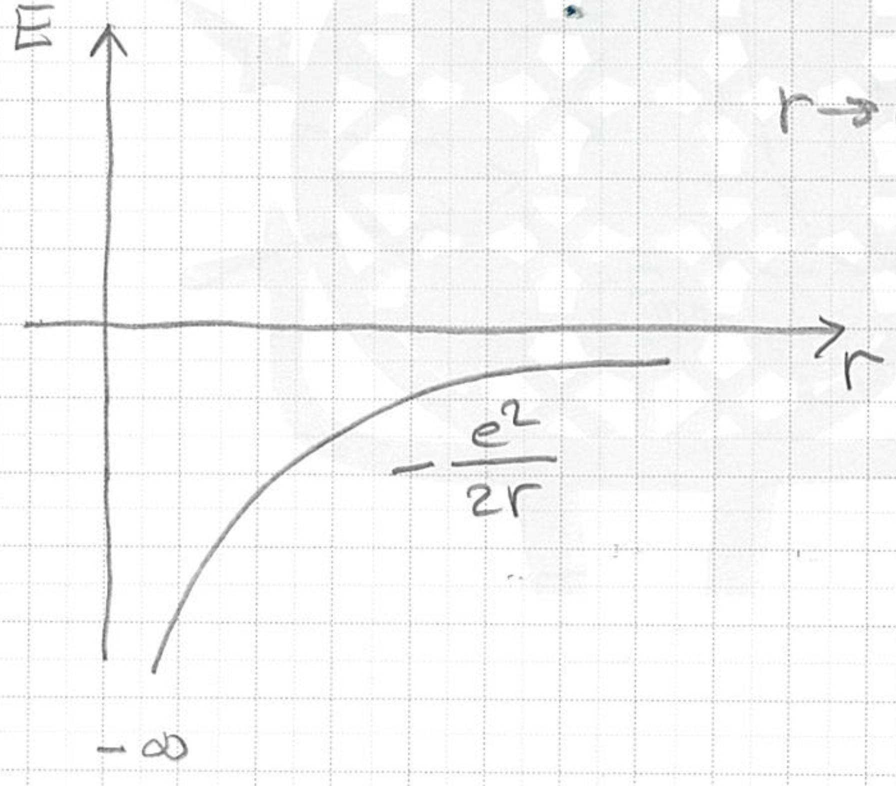
$$v = \omega r$$

$$\frac{e^2}{r^2} = \frac{m v^2}{r}$$

$$m v^2 = \frac{e^2}{r}$$

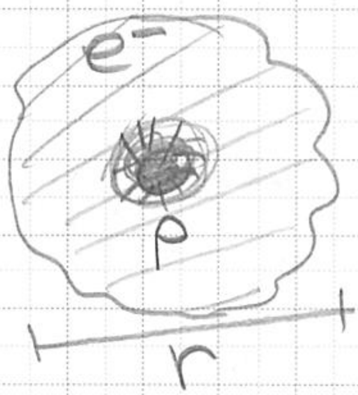
$$E = \frac{1}{2} m v^2 - \frac{e^2}{r} = \frac{e^2}{2r} - \frac{e^2}{r} = -\frac{e^2}{2r}$$

$r \to 0 \quad E \to -\infty$



Heisenberg Atomu (Kuantum Fiziği)

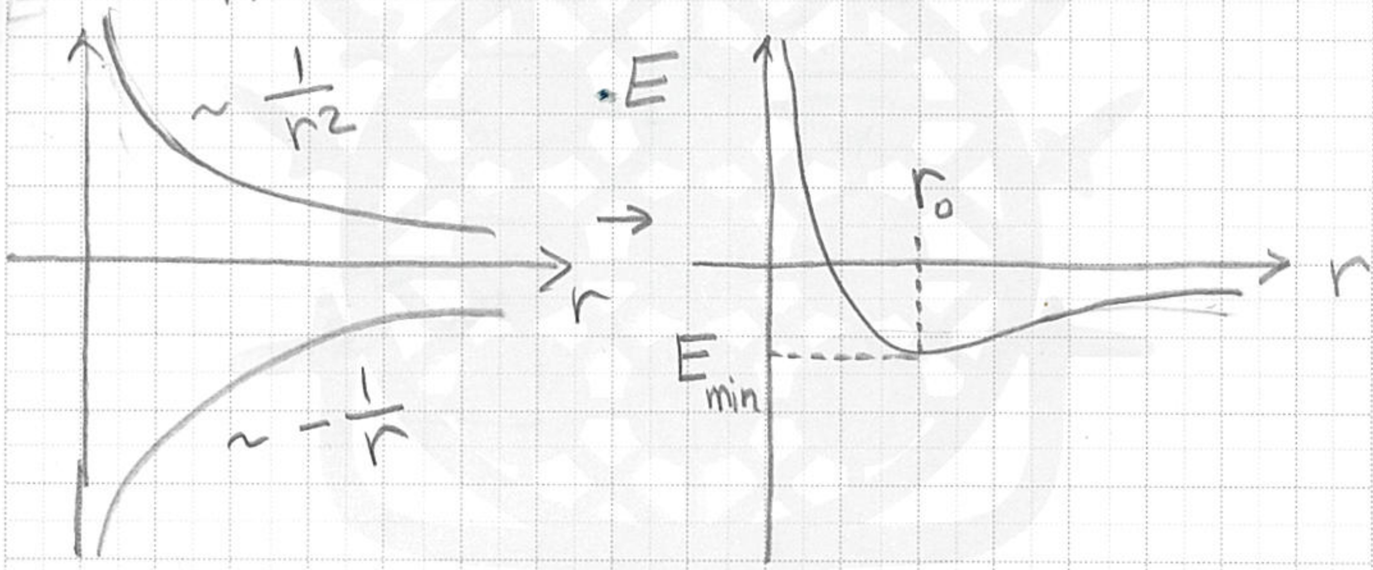
$$E = \frac{p^2}{2m} - \frac{e^2}{r}$$



$$\Delta p \, \Delta r = \frac{\hbar}{2}$$

$\Delta p \sim p \quad , \quad \Delta r \sim r$

$$p^2 = \frac{\hbar^2}{4r^2} \rightarrow E = \frac{\hbar^2}{8mr^2} - \frac{e^2}{r}$$



$$r_0 = \frac{\hbar^2}{4me^2} \qquad E_{min}(r_0) = -\frac{2me^4}{\hbar^2}$$

Harmonik Salınıcı için
Minimum (zero point) Enerji [Belirsizlik ilkesi]

$$E = \frac{p^2}{2m} + \frac{1}{2} K x^2 \qquad px \simeq \frac{\hbar}{2}$$

$$p = \frac{\hbar}{2x}$$

$$E = \frac{\hbar^2}{8mx^2} + \frac{1}{2} K x^2$$

$$\frac{dE}{dx} = 0 \qquad x_0^2 = \frac{\hbar}{2\sqrt{mK}}$$

$$E_{min} = \frac{\hbar^2}{8m} \frac{2\sqrt{mK}}{\hbar} + \frac{1}{2} K \frac{\hbar}{2\sqrt{mK}} \qquad \sqrt{\frac{K}{m}} = \omega$$

$$\boxed{E_{min} = \frac{1}{2} \hbar \omega}$$