LEGENDRE POLİNOMLARI ve FONKSİYONLARI

3.1. Legendre Denk. i ve Çözümleri:

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+ky=0$$

$k \to \ell(\ell+1)$

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\ell(\ell+1)y=0$$

$q(x)=-2x^2/(1-x^2) \quad r(x)=\left[\ell(\ell+1)x^2/(1-x^2)\right] \quad -1<x<+1$

$$z(x,s)=x^s\sum_{n=0}^{\infty}a_n x^n$$

$$a_0\, s(s-1)=0$$

$$a_1\, s(s+1)=0$$

$$a_{n+2}(s+n+2)(s+n+1)-a_n\left[(s+n)(s+n+1)-\ell(\ell+1)\right]=0$$

$s_1=0 \quad s_2=1 \quad \Delta s=1$

$s=0 \Rightarrow a_1$ belirsiz.

$s=0$'lı kök a_0 ve a_1 gibi 2 keyfi sbt'li bağımsız çözümler verir.

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$s = 0$ ile

$$a_{n+2} = a_n \frac{n(n+1) - \ell(\ell+1)}{(n+1)(n+2)}$$

$$n(n+1) - \ell(\ell+1) = n^2 + n - \ell^2 - \ell = (n^2 - \ell^2) + (n - \ell)$$

$$= (n-\ell)(n+\ell) + (n-\ell) = (n-\ell)(n+\ell+1)$$

$$\Rightarrow \quad a_{n+2} = a_n \frac{(n-\ell)(n+\ell+1)}{(n+1)(n+2)}$$

$\Downarrow$

$$a_2 = -a_0 \frac{\ell(\ell+1)}{1.2} \qquad a_3 = -a_1 \frac{(\ell-1)(\ell+2)}{2.3}$$

$$a_4 = -a_2 \frac{(\ell-2)(\ell+3)}{3.4} \qquad a_5 = -a_3 \frac{(\ell-3)(\ell+4)}{4.5}$$

$$= a_0 \frac{\ell(\ell-2)(\ell+1)(\ell+3)}{1.2.3.4} \qquad = a_1 \frac{(\ell-1)(\ell-3)(\ell+2)(\ell+4)}{2.3.4.5}$$

$$a_{2n} = (-1)^n a_0 \frac{\ell(\ell-2)(\ell-4)\cdots(\ell-2n+2)(\ell+1)(\ell+3)\cdots(\ell+2n-1)}{(2n)!}$$

$$a_{2n+1} = (-1)^n a_1 \frac{(\ell-1)(\ell-3)\cdots(\ell-2n+1)(\ell+2)(\ell+4)\cdots(\ell+2n)}{(2n+1)!}$$

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$$z(x,0) = a_0 \left\{ 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\ell(\ell-2)\cdots(\ell-2n+2)(\ell+1)(\ell+3)\cdots(\ell+2n-1)}{(2n)!} x^{2n} \right\}$$

$$+ a_1 \left\{ x + \sum_{n=1}^{\infty} (-1)^n \frac{(\ell-1)(\ell-3)\cdots(\ell-2n+1)(\ell+2)(\ell+4)\cdots(\ell+2n)}{(2n+1)!} x^{2n+1} \right\}$$

$$= a_0 y_1(x) + a_1 y_2(x)$$

y_1, y_2 $\;-1 < x < +1$ için yakınsayacaktır.

$x = \pm 1$ için yukarıdaki serinin ıraksayacağını söylüyor. $x = \pm 1$ için sonlu çözümler nasıl bulunur? Tek yolu sonsuz seriyi sonlu seriye indirgemek ile olur. Bu da ℓ'nin pozitif tam değerleri için olur.

$$\ell = 2n \Rightarrow \begin{cases} a_{2n} \neq 0 \\ a_{2n+2} = 0 \end{cases}$$

$$\ell = 2n+1 \Rightarrow \begin{cases} a_{2n+1} \neq 0 \\ a_{2n+3} \text{ ve diğer terimler} \to 0. \end{cases}$$

Legendre serilerinin yakınsaklığı.

$$\sum_{n=0}^{\infty} a_{2n} x^{2n}, \quad \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$$

$$a_{n+2} = a_n \frac{(n-\ell)(n+\ell+1)}{(n+1)(n+2)}$$

$n > \ell$ için serinin tüm terimleri aynı işarete sahip

pozitif terimli seriler için test uygulayabiliriz.

$x \neq 1 \Rightarrow$ D'Alembert oran testi

$\sum_{0}^{\infty} u_n$ pozitif terimli bir seri $\Rightarrow \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \alpha \begin{cases} \alpha < 1 & \text{ıraksak} \\ \alpha > 1 & \text{yakınsak} \end{cases}$

$$u_n = a_{2n} x^{2n}$$

$$\frac{u_n}{u_{n+1}} = \frac{a_{2n} x^{2n}}{a_{2n+2} x^{2n+2}} = \frac{(2n+1)(2n+2)}{(2n-\ell)(2n+\ell+1)} \frac{1}{x^2}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2} \begin{cases} \text{ıraksak} & x^2 > 1 & |x| > 1 \\ \text{yakınsak} & x^2 < 1 & |x| < 1 \end{cases}$$

$x = \pm 1$ için Gauss oran testi uygulanır.

$\sum_{n=0}^{\infty} u_n$ pozitif terimli seri ve $n \geq N$ = sbt.

$$\Rightarrow \frac{u_n}{u_{n+1}} = 1 + \frac{\mu}{n} + O\left(\frac{1}{n^p}\right) \quad p > 1$$

$O(1/n^p) = f(n)$ öyleki $\lim_{n\to\infty} \frac{f(n)}{1/n^p}$: sonlu

$$\Rightarrow \sum_{n=0}^{\infty} u_n : \begin{cases} \text{yakınsak} & \mu > 1 \\ \text{ıraksak} & \mu < 1 \end{cases}$$

$x = \pm 1$ için

$$\frac{u_n}{u_{n+1}} = \frac{(2n+1)(2n+2)}{(2n-\ell)(2n+\ell+1)}$$

$$= 1 + \frac{1}{n} + \frac{\ell(\ell+1)(n+1)}{[4n^2+2n-\ell(\ell+1)]n}$$

$\lim_{n\to\infty} (\;) = \frac{\ell(\ell+1)}{4}$ sonlu.

$\mu = 1$ ☹ seri ıraksak.

$x = \pm 1$ için.

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ℓ çift tam sayı $\Rightarrow y_1(x)$ } bir polinoma indirgenir.
ℓ tek tam sayı $\Rightarrow y_2(x)$ }

(#) Her iki halde de x'in en yüksek kuvveti x^ℓ'dir. Tek ve çift ℓ için geçerli bir seri elde ederiz. İlk terim $a_\ell x^\ell$ diğer terimler,

$$a_n = -a_{n+2}\frac{(n+2)(n+1)}{(\ell-n)(\ell+n+1)}$$

$$a_{\ell-2} = -a_\ell \frac{\ell(\ell-1)}{2(2\ell-1)}$$

$$a_{\ell-4} = -a_{\ell-2}\frac{(\ell-2)(\ell-3)}{4(2\ell-3)} = a_\ell \frac{\ell(\ell-1)(\ell-2)(\ell-3)}{2.4.(2\ell-1)(2\ell-3)}$$

$$a_{\ell-2r} = (-1)^r a_\ell \frac{\ell(\ell-1)(\ell-2)\ldots(\ell-2r+1)}{2.4.\ldots.2r\,(2\ell-1)(2\ell-3)\ldots(2\ell-2r+1)}$$

(o o o) ℓ : tek $\Rightarrow$ } de $y_1(x)$ ve $y_2(x)$ aynı ifadeye indirgenir.
ℓ : çift $\Rightarrow$ }

$$y(x) = a_\ell x^\ell + a_{\ell-2}x^{\ell-2} + a_{\ell-4}x^{\ell-4} + \ldots + \begin{cases} a_0 & \ell : \cancel{\text{tek}}\ \text{çift} \\ a_1 x & \ell : \cancel{\text{çift}}\ \text{tek} \end{cases}$$

$$= \sum_{r=0}^{[\ell/2]} a_{\ell-2r}\, x^{\ell-2r} \qquad \left[\frac{\ell}{2}\right] : \begin{cases} \ell/2 & \ell\ \cancel{\text{tek}}\ \text{çift} \\ (\ell-1)/2 & \ell : \text{tek} \end{cases}$$

($\ell/2$ ye eşit, daha ya da daha küçük en büyük tamsayı.)

$$= a_\ell \sum_{r=0}^{[\ell/2]} (-1)^r \frac{\ell(\ell-1)\ldots(\ell-2r+1)}{2.4.\ldots.2r\,(2\ell-1)(2\ell-3)\ldots(2\ell-2r+1)}\, x^{\ell-2r}$$

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$$\ell(\ell-1)\ldots(\ell-2r+1) = \ell(\ell-1)\ldots(\ell-2r+1)\frac{(\ell-2r)(\ell-2r-1)\ldots 3.2.1}{(\ell-2r)(\ell-2r-1)\ldots 3.2.1}$$

$$= \frac{\ell!}{(\ell-2r)!}$$

$$2.4\ldots 2r = 2^r r!$$

$$(2\ell-1)(2\ell-3)\ldots(2\ell-2r+1) = \frac{2\ell(2\ell-1)(2\ell-2)\ldots(2\ell-2r+1).(2\ell-2r)!}{2\ell(2\ell-2)(2\ell-4)\ldots(2\ell-2r+2)(2\ell-2r)!}$$

$$= \frac{(2\ell)!}{2^r\ell(\ell-1)\ldots(\ell-r+1).(2\ell-2r)!} = \frac{(2\ell)!\,(\ell-r)!}{2^r\ell!\,(2\ell-2r)!}$$

$$y(x) = a_\ell\sum_{r=0}^{[\ell/2]}(-1)^r\frac{\ell!}{(\ell-2r)!\,\cancel{2^r}r!}\frac{\cancel{2^r}(2\ell-2r)!\,\ell!}{(2\ell)!\,(\ell-r)!}x^{\ell-2r}$$

$$= a_\ell\sum_{r=0}^{[\ell/2]}(-1)^r\frac{(\ell!)^2(2\ell-2r)!}{r!\,(\ell-2r)!\,(\ell-r)!\,(2\ell)!}x^{\ell-2r}$$

$$a_\ell = \frac{(2\ell)!}{2^\ell(\ell!)^2}$$ secersek

$$P_\ell(x) = \sum_{r=0}^{[\ell/2]}(-1)^r\frac{(2\ell-2r)!}{2^\ell r!\,(\ell-r)!\,(\ell-2r)!}x^{\ell-2r} \qquad -1\leq x\leq +1$$

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2. Legendre Polinomları için Doğurucu Fonksiyon:

Teorem 1.

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{\ell=0}^{\infty} t^{\ell} P_{\ell}(x) \qquad |t|<1, \; |x|\leq 1$$

İspat: $(1-2xt+t^2)^{-1/2}$ 'yi Binom teoremine göre seriye açalım.

$$(1-2xt+t^2)^{-1/2} = \left[1-t(2x-t)\right]^{-1/2}$$

$$= 1+\left(-\frac{1}{2}\right)\left[-t(2x-t)\right]+\frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left[-t(2x-t)\right]^2$$

$$+\cdots+\frac{1}{r!}\left[\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2r-1}{2}\right)\right]\left[-t(2x-t)\right]^r+\cdots$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{1}{2^r r!}\left(1.3.5.\ldots.(2r-1)\right)(-1)^r t^r (2x-t)^r$$

$$= \sum_{r=0}^{\infty} \frac{(2r)!}{2^{2r}(r!)^2} t^r (2x-t)^r$$

$$(a+b)^n = \sum_{t=0}^{n} \binom{n}{t} a^{n-t} b^t \qquad \binom{n}{t} = \frac{n!}{t!(n-t)!}$$

Şimdi $(2x-t)^r$ 'yi Binom teoremine göre açalım.

$$(2x-t)^r = \sum_{p=0}^{r} C_p^r (2x)^{r-p}(-t)^p \quad ; \quad C_p^r = \frac{r!}{p!(r-p)!}$$

$$(1-2xt+t^2)^{-1/2} = \sum_{r=0}^{\infty}\left(\frac{(2r)!}{2^{2r}(r!)^2}\right)\sum_{p=0}^{r} C_p^r (-1)^p t^{r+p} (2x)^{r-p}$$

t^{ℓ} 'nin katsayısının P_{ℓ} olup olmadığına bakıyoruz

$r+p=\ell$ almalıyız.

$p = l - r$. Ancak $0 \leq p \leq r$ değerlerini alıyor böylece

$$0 \leq l - r \leq r$$

$\frac{l}{2} \leq r \leq l$ 'yi sağlayan r'nin değerlerini gözönüne alındıyor.

l çift $\Rightarrow$ $r : \frac{l}{2} \rightarrow l$ arasında
l tek $\Rightarrow$ $r : \frac{l+1}{2} \rightarrow l$ "
değerler alır.

r'nin bu değerlerinin herhangi birisi için t^l'nin katsayısı $p = l - r$ alarak bulunur.

$$\frac{(2r)!}{2^{2r}(r!)^2} C_{l-r}^{r} (-1)^{l-r} (2x)^{r-(l-r)}$$

toplam katsayı r üzerinden toplanarak bulunur.

$$t^l \text{'nin katsayısı} = \sum_{r=\begin{cases} \frac{l}{2} & l \text{ çift} \\ \frac{l+1}{2} & l \text{ tek} \end{cases}}^{l} \frac{(2r)!}{2^{2r}(r!)^2} C_{l-r}^{r} (-1)^{l-r} (2x)^{2r-l}$$

$r \rightarrow k = l - r$

$$= \sum_{k\begin{cases} l/2 \\ l+1/2 \end{cases}}^{0} \frac{(2l-2k)!}{2^{2l-2k}[(l-k)!]^2} C_{k}^{l-k} (-1)^k (2x)^{l-2k}$$

$$= \sum_{k=0}^{[l/2]} \frac{(2l-2k)!}{2^{2l-2k}[(l-k)!]^2} \frac{(l-k)!}{(l-2k)!\, k!} (-1)^k 2^{l-2k} x^{l-2k}$$

$$= \sum_{k=0}^{[l/2]} (-1)^k \frac{(2l-2k)!}{2^l (l-k)!(l-2k)!\, k!} x^{l-2k} = P_l(x)$$

3. Legendre Polinomları İçin Diğer İfadeler:

Teorem 2. (Rodrigues Formülü)

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2-1)^\ell$$

İspat: $(x^2-1)^\ell$'yi Binom teoremine göre açalım.

$$(x^2-1)^\ell = \sum_{r=0}^{\ell} C_\ell^r (-1)^r x^{2(\ell-r)}$$

$$\Rightarrow \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2-1)^\ell = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} \sum_{r=0}^{\ell} C_\ell^r (-1)^r x^{2(\ell-r)}$$

x'in ℓ'den küçük kuvvetinin türevi sıfırdır;

$$\frac{d^\ell}{dx^\ell} x^{2\ell-2r} = 0 \qquad 2\ell-2r<\ell \text{ yani } r>\ell/2 \text{ ise}$$

$\sum_{r=0}^{\ell}$'yi $\left.\begin{array}{l} \ell \text{ çift için } \sum_{r=1}^{\ell/2} \\ \ell \text{ tek için } \sum_{r=0}^{(\ell-1)/2} \end{array}\right\}$ ile yani $\sum_{r=0}^{[\ell/2]}$ ile değiştirelim.

$$\frac{d^\ell}{dx^\ell} x^p = p(p-1)(p-2)\dots(p-(\ell+1))\,x^{p-\ell}$$

$$= \frac{p!}{(p-\ell)!} x^{p-\ell}$$

$$\Rightarrow \frac{d^\ell}{dx^\ell} x^{2(\ell-r)} = \frac{(2\ell-2r)!}{(\ell-2r)!} x^{\ell-2r}$$

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$$\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l = \frac{1}{2^l \cancel{l!}}\sum_{r=0}^{[l/2]}\frac{\cancel{l!}}{r!\,(l-r)!}(-1)^r\frac{(2l-2r)!}{(l-2r)!}x^{l-2r}$$

$$= \sum_{r=0}^{[l/2]}(-1)^r\frac{(2l-2r)!}{2^l\, r!\,(l-r)!\,(l-2r)!}x^{l-2r}$$

$$= P_l(x)$$

Teorem 3. Laplace İntegral Gösterimi :

$$P_l(x) = \frac{1}{\pi}\int_0^{\pi}\left[x+\sqrt{x^2-1}\cos\theta\right]^l d\theta$$

İspat : $$\int_0^{\pi}\frac{d\theta}{1+\lambda\cos\theta} = \frac{\pi}{\sqrt{1-\lambda^2}} \qquad t = \operatorname{tg}\frac{\theta}{2}$$

$$\lambda = -\frac{\left[u\sqrt{x^2-1}\right]}{1-ux}$$ koyup u'ya göre seriye açalım;

$1+\cancel{u^2x^2}-2ux-\cancel{u^2x^2}+u^2$

$$\frac{1}{1+\lambda\cos\theta} = \frac{1}{1-\frac{u\sqrt{x^2-1}}{1-ux}\cos\theta}$$

$$= (1-ux)\left[1-u\left[x+\sqrt{x^2-1}\cos\theta\right]\right]^{-1}$$

$$= (1-ux)\sum_{l=0}^{\infty}u^l\left[x+\sqrt{x^2-1}\cos\theta\right]^l$$

$$\frac{1}{1-a} = \sum_{n=0}^{\infty}a^n$$

$$\frac{1}{\sqrt{1-\lambda^2}} = \frac{1}{\sqrt{1-\frac{u^2(x^2-1)}{(1-ux)^2}}} = \frac{1-ux}{\sqrt{(1-ux)^2-u^2(x^2-1)}} = \frac{1-ux}{\sqrt{1-2ux+u^2}}$$

$$\Rightarrow \int_0^{\pi}\sum_{l=0}^{\infty}u^l\left[x+\left[\sqrt{x^2-1}\right]\cos\theta\right]^l d\theta = \frac{\pi}{\sqrt{1-2ux+u^2}}$$

$$\sum_{\ell=0}^{\infty} u^\ell \int_0^\pi \{x+\sqrt{x^2-1}\cos\theta\}^\ell d\theta = \pi \sum_{\ell=0}^{\infty} u^\ell P_\ell(x)$$

$$\Rightarrow \pi P_\ell(x) = \int_0^\pi \{x+\sqrt{x^2-1}\cos\theta\}^\ell d\theta$$

4. Legendre Polinomlarının Özel Değerleri

$P_0(x) = 1$ $\qquad$ $P_3(x) = \frac{1}{2}(5x^3-3x)$

$P_1(x) = x$ $\qquad$ $P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$

$P_2(x) = \frac{1}{2}(3x^2-1)$

Teorem 4. (i) $P_\ell(1) = 1$ (ii) $P_\ell(-1) = (-1)^\ell$

(iii) $P_\ell'(1) = \frac{1}{2}\ell(\ell+1)$ (iv) $P_\ell'(-1) = (-1)^{\ell-1}\frac{1}{2}\ell(\ell+1)$

(v) $P_{2\ell}(0) = (-1)^\ell \dfrac{(2\ell)!}{2^{2\ell}(\ell!)^2}$ (vi) $P_{2\ell+1}(0) = 0$

İspat: (i) $\dfrac{1}{\sqrt{1-2xt+t^2}} = \sum_{\ell=0}^{\infty} t^\ell P_\ell(x)$

$x=1 \Rightarrow \dfrac{1}{1-t} = \sum_{\ell=0}^{\infty} t^\ell P_\ell(1) = \sum_{\ell=0}^{\infty} t^\ell$

$(\Rightarrow) \; P_\ell(1) = 1$

(ii) $x=-1$ önceki gibi

(iii) $P_\ell(x)$ Legendre diff. denk.-ini sağlar.

$$(1-x^2)P_\ell''(x) - 2xP_\ell'(x) + \ell(\ell+1)P_\ell(x) = 0$$

$x=1 \Rightarrow -2P_\ell'(1) + \ell(\ell+1)P_\ell(1) = 0$

$$P_\ell'(1) = \frac{1}{2}\ell(\ell+1)$$

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(iv) önceki gibi, $x=-1$ alarak,

(v, vi) $$(1-2xt+t^2)^{-1/2} = \sum_{\ell=0}^{\infty} t^\ell P_\ell(x)$$

$$x=0 \Rightarrow (1+t^2)^{-1/2} = \sum_{\ell=0}^{\infty} t^\ell P_\ell(0)$$

$$(1+t^2)^{-1/2} = 1 + \left(-\frac{1}{2}\right)t^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(t^2)^2 + \cdots$$
$$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left[-(2\ell-1)/2\right]}{\ell!}(t^2)^\ell + \cdots$$
$$= \sum_{\ell=0}^{\infty} (-1)^\ell \frac{1.3.5\ldots(2\ell-1)}{2^\ell\,\ell!} t^{2\ell}$$
$$= \sum_{\ell=0}^{\infty} (-1)^\ell \frac{1.2.3.4\ldots(2\ell-2)(2\ell-1)2\ell}{2^\ell\,\ell!\;2.4.6\ldots 2\ell} t^{2\ell}$$
$$= \sum_{\ell=0}^{\infty} (-1)^\ell \frac{(2\ell)!}{2^{2\ell}(\ell!)^2} t^{2\ell} = \sum_{\ell=0}^{\infty} \cancel{(-1)^\ell\, t^{2\ell}}$$

$$\Rightarrow \sum (-1)^\ell \frac{(2\ell)!}{2^{2\ell}(\ell!)^2} t^{2\ell} = \sum_{\ell=0}^{\infty} t^\ell P_\ell(0)$$

$$\begin{cases} P_{2\ell}(0) = (-1)^\ell \dfrac{(2\ell)!}{2^{2\ell}(\ell!)^2} \\ P_{2\ell+1}(0) = 0 \end{cases}$$

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4.5. Legendre Polinomlarının Dikliği.

Teorem 5. $\displaystyle\int_{-1}^{+1} P_\ell(x)\,P_m(x)\,dx = \frac{2}{2\ell+1}\,\delta_{\ell m}$

İspat:

$$\begin{cases} (1-x^2)P_\ell'' - 2xP_\ell' + \ell(\ell+1)P_\ell = 0 \\ (1-x^2)P_m'' - 2xP_m' + m(m+1)P_m = 0 \end{cases}$$

$$\begin{array}{l} P_m\ / \\ P_\ell\ / \end{array}\begin{cases} \dfrac{d}{dx}\left\{(1-x^2)P_\ell'\right\} + \ell(\ell+1)P_\ell = 0 \\ \dfrac{d}{dx}\left\{(1-x^2)P_m'\right\} + m(m+1)P_m = 0 \end{cases}$$ şeklinde yazılabilirler (S.L)

$$\int_{-1}^{+1}\left[P_m\frac{d}{dx}\left\{(1-x^2)P_\ell'\right\} - P_\ell\frac{d}{dx}\left\{(1-x^2)P_m'\right\}\right]dx$$
$$+\ \left\{\ell(\ell+1) - m(m+1)\right\}\int_{-1}^{+1} P_\ell P_m\,dx = 0$$

(*) $$\frac{d}{dx}\left\{P_m(1-x^2)\frac{dP_\ell}{dx}\right\} = \frac{dP_m}{dx}(1-x^2)\frac{dP_\ell}{dx} + P_m\frac{d}{dx}\left\{(1-x^2)\frac{dP_\ell}{dx}\right\}$$

$$\Rightarrow \int_{-1}^{+1}\left[\frac{d}{dx}\left\{P_m(1-x^2)P_\ell'\right\} - \cancel{P_m'(1-x^2)P_\ell'} - \frac{d}{dx}\left\{P_\ell(1-x^2)P_m'\right\}\right.$$
$$\left. + \cancel{P_\ell'(1-x^2)P_m'}\right]dx + (\ell^2+\ell-m^2-m)\int_{-1}^{+1} P_\ell P_m\,dx = 0$$

$$\Big[\underbrace{P_m(1-x^2)P_\ell' - P_\ell(1-x^2)P_m'}_{0}\Big]_{-1}^{+1} + (\ell-m)(\ell+m+1)\int_{-1}^{+1} P_\ell P_m\,dx = 0$$

$$\ell \neq m \Rightarrow \int_{-1}^{+1} P_\ell P_m\,dx = 0$$

$$\int_{-1}^{+1} [P_\ell(x)]^2 dx = \frac{2}{2\ell+1}$$

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{\ell=0}^{\infty} t^\ell P_\ell(x)$$

$$\Rightarrow \frac{1}{1-2xt+t^2} = \left\{\sum_{\ell=0}^{\infty} t^\ell P_\ell(x)\right\}^2 = \sum_{\ell=0}^{\infty} t^\ell P_\ell(x) \sum_{m=0}^{\infty} t^m P_m(x)$$

$$= \sum_{\ell,m=0}^{\infty} t^{\ell+m} P_\ell(x) P_m(x)$$

$$\int \frac{dx}{a+bx} = \frac{1}{b}\ln(a+bx)$$

$$\Rightarrow \int_{-1}^{+1} \frac{dx}{(1-2xt+t^2)} = \sum_{\ell,m} t^{\ell+m} \int_{-1}^{+1} P_\ell(x) P_m(x)\, dx$$

$$-\frac{1}{2t}\ln(1+t^2-2tx)\Big|_{-1}^{+1} = \sum_{\ell=0}^{\infty} t^{2\ell} \int_{-1}^{+1} \{P_\ell(x)\}^2 dx$$

$$\Rightarrow \sum_\ell t^{2\ell} \int_{-1}^{+1} [P_\ell(x)]^2 dx = -\frac{1}{2t}\ln(1+t^2-2t) + \frac{1}{2t}\ln(1+t^2+2t)$$

$$= \frac{1}{t}\ln(1+t) - \frac{1}{t}\ln(1-t)$$

$$= \frac{1}{t}\left\{t - \frac{t^2}{2} + \frac{t^3}{3} + \cdots + t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots\right\}$$

$$= \frac{1}{t}\left\{2t + 2\frac{t^3}{3} + 2\frac{t^5}{5} + \cdots\right\}$$

$$= 2\left\{1 + \frac{t^2}{3} + \frac{t^4}{5} + \cdots\right\}$$

$$= 2\sum_{\ell=0}^{\infty} \frac{t^{2\ell}}{2\ell+1}$$

$$\Rightarrow \int_{-1}^{+1} \{P_\ell(x)\}^2 dx = \frac{2}{2\ell+1}$$

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4.6. Legendre Serileri

Teorem 6. $f(x)$ n. dereceden bir polinom ise,

$$f(x)=\sum_{r=0}^{n} C_r P_r(x) \quad , \quad C_r=\left(r+\frac{1}{2}\right)\int_{-1}^{+1} f(x) P_r(x)\,dx$$

Eğer $f(x)$ çift (tek) ise, yalnızca C_r'nin çift (tek) indisli olanları sıfırdan farklıdır.

İspat:

$$f(x)=a_n x^n+a_{n-1}x^{n-1}+\dots+a_1 x+a_0 \quad \text{olsun,}$$

$$P_n(x)=k_n x^n+k_{n-2}x^{n-2}+\dots+$$

$f(x)-\left(\frac{a_n}{k_n}\right)P_n(x)$ alırsak, ya sıfır ya da $(n-1)$. dereceden başka bir polinom elde ederiz.

$$f(x)=C_n P_n(x)+g_{n-1}(x) \qquad C_n=\frac{a_n}{k_n}$$

$n-1$. dereceden polinom.

Aynı tartışmayı g_{n-1}'e uygularız:

$$f(x)=C_n P_n(x)+C_{n-1}P_{n-1}(x)+g_{n-2}(x)$$

$$=C_n P_n(x)+C_{n-1}P_{n-1}(x)+\dots+C_1 P_1(x)+C_0 P_0(x)$$

$$=\sum_{r=0}^{n} C_r P_r(x)$$

$$\Rightarrow \int_{-1}^{+1} f(x) P_s(x)\,dx=\sum_{r=0}^{n} C_r \int_{-1}^{+1} P_r(x) P_s(x)\,dx$$

$$=C_s \frac{2}{2s+1}$$

$$\Rightarrow C_r=\left(r+\frac{1}{2}\right)\int_{-1}^{+1} f(x) P_r(x)\,dx$$

$f(x)$ çift olsun.

r çift ise P_r çift
r tek ise P_r tek olduğundan

r çift ise fP_r çift
r tek ise fP_r tek $\int_{-1}^{+1}=0$

r tek ise $C_r=0$

Sonuç: $f(x)$, ℓ'den daha düşük dereceli bir polinom ise

$$\int_{-1}^{+1} f(x) P_\ell(x)\, dx = 0$$

İspat: $f(x)$ n. dereceden olsun.

$$f(x) = \sum_{r=0}^{n} c_r P_r(x)$$

$$\Rightarrow \int_{-1}^{+1} f(x) P_\ell(x)\, dx = \sum_{r=0}^{n} c_r \int_{-1}^{+1} P_\ell(x) P_r(x)\, dx = 0$$

çünkü $r \leq n \leq \ell$ öyleki r hiçbir zaman ℓ'ye eşit olmaz

Teorem 7. $f(x)$ $-1 \leq x \leq +1$ aralığında aşağıdaki koşulları sağlasın.

(i) $f(x)$ parçalı sürekli (sonlu sayıda sonlu süreksizlikler ~~durumunda~~ dışında sürekli)

(ii) $f(x)$ 'in sonlu sayıda max. ve min.'u bulunsun.

O zaman $c_r = (r + \frac{1}{2}) \int_{-1}^{+1} f(x) P_r(x)\, dx$ olmak üzere

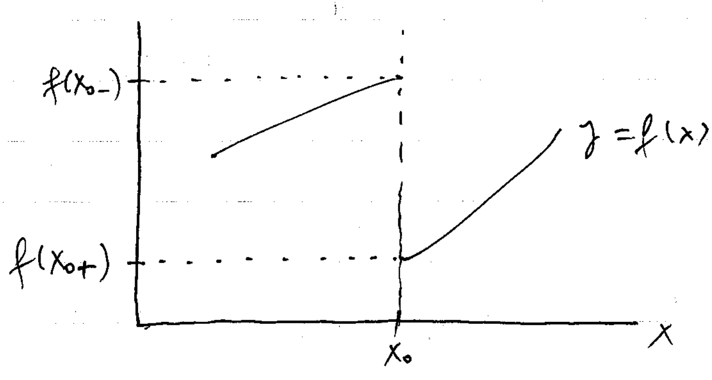
(x $f(x)$'in süreksizlik noktası değil ise) $\sum_r c_r P_r(x)$ serisi $f(x)$'e yakınsar.

x $f(x)$'in süreksizlik noktası ise

$$\frac{1}{2}\{f(x_+) + f(x_-)\}$$

'e yakınsar. Uç noktalarında seri karşılıklı olarak ($x = \pm 1$) $f(1_-)$ ve $f(1_+)$ 'ya yakınsar.

Bu seri $f(x)$ için Legendre serisi olarak adlandırılır.



BS K

4.7. TÜRETME BAĞINTILARI.

Teorem 8.

(i) $P_\ell'(x) = \sum_{r=0}^{[\ell-1/2]} (2\ell-4r-1)\, P_{\ell-2r-1}(x)$

(ii) $x P_\ell(x) = \frac{\ell+1}{2\ell+1} P_{\ell+1}(x) + \frac{\ell}{2\ell+1} P_{\ell-1}(x)$

(iii) $(\ell+1)P_{\ell+1}(x) - (2\ell+1)x P_\ell(x) + \ell P_{\ell-1}(x) = 0$

(iv) $P_{\ell+1}'(x) - P_{\ell-1}'(x) = (2\ell+1)P_\ell(x)$

(v) $x P_\ell'(x) - P_{\ell-1}'(x) = \ell P_\ell(x)$

(vi) $P_\ell'(x) - x P_{\ell-1}'(x) = \ell P_{\ell-1}(x)$

(vii) $(x^2-1) P_\ell'(x) = \ell x P_\ell(x) - \ell P_{\ell-1}(x)$

(viii) $(x^2-1) P_\ell'(x) = (\ell+1) P_{\ell+1}(x) - (\ell+1) x P_\ell(x)$

(ix) $\sum_{k=0}^{\ell} (2k+1) P_k(x) P_k(y) = \frac{\ell+1}{x-y}\left\{ P_{\ell+1}(x)P_\ell(y) - P_\ell(x)P_{\ell+1}(y)\right\}$

İspat:

(i) $P_\ell(x)$: ℓ. dereceden bir polinom ; çift ℓ x'in çift kuvvetleri, tek ℓ, x'in tek kuvvetleri.

$\Rightarrow$ $P_\ell'(x)$: $(\ell-1)$. dereceden bir polinom.'dur.

$$P_\ell'(x) = C_{\ell-1}P_{\ell-1}(x) + C_{\ell-3}P_{\ell-3}(x) + \dots + C_{\ell-2r-1}P_{\ell-2r-1}(x) + \dots$$

$$+ \begin{cases} C_1 P_1(x) & \ell : \text{çift} \\ C_0 P_0(x) & \ell : \text{tek} \end{cases}$$

$$C_s = \left(s+\tfrac{1}{2}\right) \int_{-1}^{+1} P_\ell'(x)\, P_s(x)\, dx$$

$$= \left(s+\tfrac{1}{2}\right) \left\{ \Big[P_\ell(x) P_s(x) \Big]_{-1}^{+1} - \cancel{\int_{-1}^{+1} P_\ell(x) P_s'(x)\, dx} \right\} \to 0$$

$(s-1)$. mertebeden ve $s-1$ daima ℓ'den küçük

$$= \left(s+\tfrac{1}{2}\right) \left\{ P_\ell(1) P_s(1) - P_\ell(-1) P_s(-1) \right\}$$

$$C_s = \left(s+\tfrac{1}{2}\right) \left\{ 1 - (-1)^{\ell+s} \right\}$$

s: $\ell-1$, $\ell-3$, ... değerlerini alır

$\Rightarrow$ $s+\ell \to 2\ell-1,\ 2\ell-3, \ldots$ //.

$$(-1)^{s+\ell} = -1$$

$$\Rightarrow \quad C_s = \left(s+\tfrac{1}{2}\right) 2 = 2s+1$$

$$C_{\ell-2r-1} = 2\,(\ell-2r-1)+1 = 2\ell-4r-1$$

$$P_\ell'(x) = (2\ell-1) P_{\ell-1}(x) + (2\ell-5) P_{\ell-3}(x) + \cdots$$

$$+ (2\ell-4r-1) P_{\ell-2r-1}(x) + \cdots \begin{cases} 3P_1(x) & \text{çift} \\ P_0(x) & \text{tek.} \end{cases}$$

$$= \sum_{r=0}^{\left[\frac{\ell-1}{2}\right]} (2\ell-4r-1) P_{\ell-2r-1}(x) \qquad \sum_{r=0} C_{\ell-2r-1} P_{\ell-2r-1}$$

$$P_\ell' = \sum_{s=0}^{\ell-1} C_s P_s(x) \qquad C_s = \left(s+\tfrac{1}{2}\right) \int_{-1}^{+1} P_\ell' P_s$$

$$P_\ell' = C_{\ell-1} P_{\ell-1} + C_{\ell-3} P_{\ell-3} + \cdots + \begin{cases} C_1 P_1 & \ell \text{ çift} \\ C_0 P_0 & \ell \text{ tek.} \end{cases}$$

(ii) $x P_\ell(x) = \frac{\ell+1}{2\ell+1} P_{\ell+1}(x) + \frac{\ell}{2\ell+1} P_{\ell-1}(x)$

$x P_\ell(x)$: $(\ell+1)$. dereceden bir polinom. ℓ çift ise tek.

ℓ tek ise çift.

$$x P_\ell(x) = C_{\ell+1} P_{\ell+1}(x) + C_{\ell-1} P_{\ell-1}(x) + \dots + \begin{cases} C_1 P_1(x) & \ell: \text{çift} \\ C_0 P_0(x) & \ell: \text{tek} \end{cases}$$

$$C_r = \left(r+\frac{1}{2}\right) \int_{-1}^{+1} x P_\ell(x) P_r(x)\, dx$$

$$= \left(r+\frac{1}{2}\right) \int_{-1}^{+1} P_\ell(x) \left\{ x P_r(x) \right\} dx$$

$r+1 < \ell$ ise integral sıfır.

$$x P_\ell(x) = C_{\ell+1} P_{\ell+1}(x) + C_{\ell-1} P_{\ell-1}(x)$$

x. e göre türetelim.

$$P_\ell(x) + x P'_\ell(x) = C_{\ell+1} P'_{\ell+1}(x) + C_{\ell-1} P'_{\ell-1}(x)$$

$$x=1 \begin{cases} P_\ell(1) = C_{\ell+1} P_{\ell+1}(1) + C_{\ell-1} P_{\ell-1}(1) \\ P_\ell(1) + P'_\ell(1) = C_{\ell+1} P'_{\ell+1}(1) + C_{\ell-1} P'_{\ell-1}(1) \end{cases}$$

$$1 = C_{\ell+1} + C_{\ell-1}$$

$$1 + \frac{1}{2}\ell(\ell+1) = C_{\ell+1}\left\{\frac{1}{2}(\ell+1)(\ell+2)\right\} + C_{\ell-1}\left\{\frac{1}{2}\ell(\ell-1)\right\}$$

$$C_{\ell+1} = \frac{\ell+1}{2\ell+1} \qquad C_{\ell-1} = \frac{\ell}{2\ell+1}$$

$$x P_\ell = \frac{\ell+1}{2\ell+1} P_{\ell+1} + \frac{\ell}{2\ell+1} P_{\ell-1}$$

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(iii) $(\ell+1)P_{\ell+1}(x) - (2\ell+1)xP_\ell(x) + \ell P_{\ell-1}(x) = 0$

(iv) (i) 'nin sonucunu $P'_{\ell+1}$ ve $P'_{\ell-1}$ için kullanalım.

$P_\ell'(x) = (2\ell-1)P_{\ell-1} + (2\ell-5)P_{\ell-3} + \cdots + (2\ell-4r-1)P_{\ell-2r-1} + \cdots \begin{cases} 3P_1 \\ P_0 \end{cases}$

$P'_{\ell+1}(x) = (2\ell+1)P_\ell(x) + (2\ell-3)P_{\ell-2}(x) + (2\ell-7)P_{\ell-4}(x) + \cdots$

$P'_{\ell-1}(x) = \qquad\qquad + (2\ell-3)P_{\ell-2}(x) + (2\ell-7)P_{\ell-4}(x) + \cdots$

$$\boxed{P'_{\ell+1} - P'_{\ell-1} = (2\ell+1)P_\ell}$$

(v) $(\ell+1)P_{\ell+1} - (2\ell+1)xP_\ell + \ell P_{\ell-1} = 0$ (iii)

* x'e göre türet.

$(\ell+1)P'_{\ell+1} - (2\ell+1)\{P_\ell + xP_\ell'\} + \ell P'_{\ell-1} = 0$

(iv) den $P'_{\ell+1}$ 'i kullanalım.

$(\ell+1)\{P'_{\ell-1}(x) + (2\ell+1)P_\ell\} - (2\ell+1)\{P_\ell + xP_\ell'\} + \ell P'_{\ell-1} = 0$

$(2\ell+1)P'_{\ell-1} + (2\ell+1)(\ell+1-1)P_\ell - (2\ell+1)xP_\ell' = 0$

$P'_{\ell-1} + \ell P_\ell - xP_\ell' = 0$

$$\boxed{xP_\ell' - P'_{\ell-1} = \ell P_\ell}$$

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(vi) $P_{\ell+1}' - P_{\ell-1}' = (2\ell+1)P_\ell$ (iv) ✓

$$xP_{\ell+1}' - xP_{\ell-1}' = (2\ell+1)xP_\ell$$

(iii) → $(\ell+1)P_{\ell+1} - (2\ell+1)xP_\ell + \ell P_{\ell-1} = 0$

$$\rightarrow (\ell+1)P_{\ell+1} + \ell P_{\ell-1} = xP_{\ell+1}' - xP_{\ell-1}'$$

$$\begin{cases} xP_\ell' - P_{\ell-1}' = \ell P_\ell & \text{(v) den.} \\ \ell \to \ell+1 \quad xP_{\ell+1}' - P_\ell' = (\ell+1)P_{\ell+1} \end{cases}$$

$$\cancel{xP_{\ell+1}'} - P_\ell' + \ell P_{\ell-1} = \cancel{xP_{\ell+1}'} - xP_{\ell-1}'$$

$$P_\ell' - xP_{\ell-1}' = \ell P_{\ell-1}$$

(vii) (v) 'den $xP_\ell' - P_{\ell-1}' = \ell P_\ell$

x ile çarp: $x^2P_\ell' - xP_{\ell-1}' = x\ell P_\ell$

(vi) 'dan $P_\ell' - xP_{\ell-1}' = \ell P_{\ell-1}$

$-$

$$(x^2-1)P_\ell' = \ell xP_\ell - \ell P_{\ell-1}$$

(viii) (v) ve (vi) da $\ell \to \ell+1$ al.

$$xP_{\ell+1}' - P_\ell' = (\ell+1)P_{\ell+1}$$

$$P_{\ell+1}' - xP_\ell' = (\ell+1)P_\ell$$ / x ile çarp farkını al.

$$-P_\ell' + x^2P_\ell' = (\ell+1)P_{\ell+1} - (\ell+1)xP_\ell$$

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$$(x^2-1)P'_\ell = (\ell+1)P_{\ell+1} - (\ell+1)xP_\ell$$

(ix)

$$\text{(iii)} \to \quad P_{k+1} = \{(2k+1)xP_k - kP_{k-1}\}\frac{1}{k+1}$$

$$P_{k+1}(x)P_k(y) - P_k(x)P_{k+1}(y)$$

$$= \frac{1}{k+1}\Big[\{(2k+1)xP_k(x) - kP_{k-1}(x)\}P_k(y)$$

$$- \{(2k+1)yP_k(y) - kP_{k-1}(y)\}P_k(x)\Big]$$

$$= \frac{2k+1}{k+1}(x-y)P_k(x)P_k(y) + \frac{k}{k+1}\Big[P_k(x)P_{k-1}(y) - P_{k-1}(x)P_k(y)\Big]$$

$$(k+1)\{P_{k+1}(x)P_k(y) - P_k(x)P_{k+1}(y)\}$$

$$= (2k+1)(x-y)P_k(x)P_k(y) + k\Big[P_k(x)P_{k-1}(y) - P_{k-1}(x)P_k(y)\Big]$$

$(k+1)[P_{k+1}(x)P_k(y) - P_k(x)P_{k+1}(y)] = f_k$ diyelim.

(*) $\quad f_k = (2k+1)(x-y)P_k(x)P_k(y) + f_{k-1} \qquad k \geq 1$ için doğru.

$k=0$'ı katmak için $f_{-1}=0$ olarak tanımlarız.

$$f_0 = P_1(x)P_0(y) - P_0(x)P_1(y)$$

$$= x-y$$

(*)'da ($k=0$ alırsak $f_0 = x-y$ gelmeli)

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(*) denk.ni $k=0$'dan $k=\ell$'ye kadar toplayalım.

$$\sum_{k=0}^{\ell} f_k = \sum_{k=0}^{\ell} (2k+1)\,(x-y)\,P_k(x)\,P_k(y) + \sum_{k=0}^{\ell} f_{k-1}$$

$$= \sum_{k=0}^{\ell} (2k+1)\,(x-y)\,P_k(x)P_k(y) + \sum_{k=1}^{\ell} f_{k-1}$$

$$= \sum_{k=0}^{\ell} (2k+1)\,(x-y)P_k(x)\,P_k(y) + \sum_{k=0}^{\ell-1} f_k$$

$$\sum_{k=0}^{\ell} f_k - \sum_{k=0}^{\ell-1} f_k = \sum_{k=0}^{\ell} (2k+1)\,(x-y)\,P_k(x)P_k(y)$$

$$f_\ell = (x-y)\sum_{k=0}^{\ell} (2k+1)\,P_k(x)\,P_k(y)$$

$$\left(\frac{\ell+1}{x-y}\right)\left[\,P_{\ell+1}(x)\,P_\ell(y) - P_\ell(x)\,P_{\ell+1}(y)\,\right] = \sum_{k=0}^{\ell} (2k+1)\,P_k(x)\,P_k(y)$$

BS K

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3.8. BİRLEŞİK LEGENDRE FONKSİYONLARI

Teorem 3.9.

Eğer z,

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \ell(\ell+1)y = 0$$

denkleminin çözümü ise

$$(1-x^2)^{m/2}\frac{d^m z}{dx^m}$$

$$(1-x^2)y'' - 2xy' + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]y = 0$$

denkleminin bir çözümüdür.

İspat:

$$(1-x^2)\frac{d^2z}{dx^2} - 2x\frac{dz}{dx} + \ell(\ell+1)z = 0$$

denk.ini x'e göre m defa türetelim.

$$\frac{d^m}{dx^m}\left\{(1-x^2)\frac{d^2z}{dx^2}\right\} - 2\frac{d^m}{dx^m}\left\{x\frac{dz}{dx}\right\} + \ell(\ell+1)\frac{d^m z}{dx^m} = 0$$

Leibniz teoremi

$$\frac{d^m}{dx^m}(uv) = \sum_{r=0}^{m}\binom{m}{r}\frac{d^r u}{dx^r}\frac{d^{m-r}v}{dx^{m-r}}$$

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$$(1-x^2)\frac{d^{m+2}}{dx^{m+2}}z + m\frac{d}{dx}(1-x^2)\frac{d^{m+1}}{dx^{m+1}}z + \frac{m(m-1)}{2}\frac{d^2}{dx^2}(1-x^2)\frac{d^m}{dx^m}z$$

$$-2\left\{x\frac{d^{m+1}}{dx^{m+1}}z + m\frac{d}{dx}x\cdot\frac{d^m z}{dx^m}\right\} + \ell(\ell+1)\frac{d^m z}{dx^m} = 0$$

$$(1-x^2)\frac{d^{m+2}}{dx^{m+2}}z - 2x(m+1)\frac{d^{m+1}}{dx^{m+1}}z + \left\{(\ell)(\ell+1) - m(m-1) - 2m\right\}\frac{d^m}{dx^m}z = 0$$

$$\frac{d^m}{dx^m}z = z_1$$

$$(1-x^2)\frac{d^2 z_1}{dx^2} - 2(m+1)x\frac{dz_1}{dx} + \left[\ell(\ell+1) - m(m+1)\right]z_1 = 0$$

$$z_2 = (1-x^2)^{m/2}z_1 = (1-x^2)^{m/2}\frac{d^m}{dx^m}z$$

$$(1-x^2)\frac{d^2}{dx^2}\left\{z_2(1-x^2)^{-m/2}\right\} - 2(m+1)x\frac{d}{dx}\left\{z_2(1-x^2)^{-m/2}\right\}$$

$$+\left\{\ell(\ell+1) - m(m+1)\right\}z_2(1-x^2)^{-m/2} = 0$$

Şimdi ele alalım:

$$\frac{d}{dx}\left[z_2(1-x^2)^{-m/2}\right] = \frac{dz_2}{dx}(1-x^2)^{-m/2}$$

$$+ z_2\left(-\frac{m}{2}\right)(1-x^2)^{-\frac{m}{2}-1}(-2x)$$

$$= \frac{dz_2}{dx}(1-x^2)^{-m/2} + m z_2 x(1-x^2)^{-\frac{m}{2}-1}$$

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$$\frac{d^2}{dx^2}\left[z_2(1-x^2)^{-m/2}\right] = \frac{d^2z_2}{dx^2}(1-x^2)^{-m/2} + \frac{dz_2}{dx}\left(-\frac{m}{2}\right)(1-x^2)^{-\frac{m}{2}-1}(-2x)$$

$$+ m\left\{\frac{dz_2}{dx}x(1-x^2)^{-\frac{m}{2}-1} + z_2(1-x^2)^{-\frac{m}{2}-1}\right.$$

$$\left. + z_2x\left(-\frac{m}{2}-1\right)(1-x^2)^{-\frac{m}{2}-2}(-2x)\right\}$$

$$= \frac{d^2z_2}{dx^2}(1-x^2)^{-m/2} + \frac{dz_2}{dx}mx(1-x^2)^{-\frac{m}{2}-1}$$

$$+ m\frac{dz_2}{dx}x(1-x^2)^{-\frac{m}{2}-1} + mz_2(1-x^2)^{-\frac{m}{2}-1}$$

$$+ mz_2x^2(m+2)(1-x^2)^{-\frac{m}{2}-2}$$

(*) $$\frac{d^2z_2}{dx^2}(1-x^2)^{-\frac{m}{2}+1} + 2mx(1-x^2)^{-\frac{m}{2}}\frac{dz_2}{dx} + mz_2(1-x^2)^{-m/2}$$

$$+ m(m+2)(1-x^2)^{-\frac{m}{2}-1}x^2z_2$$

$$- 2(m+1)x\left\{(1-x^2)^{-m/2}\frac{dz_2}{dx} + mx(1-x^2)^{-\frac{m}{2}-1}z_2\right\}$$

$$+\left[l(l+1) - m(m+1)\right]z_2(1-x^2)^{-m/2} = 0$$

$(1-x^2)^{-m/2}$ ile böl

$$(1-x^2)\frac{d^2z_2}{dx^2} + \left[2mx - 2(m+1)x\right]\frac{dz_2}{dx}$$

$$+\left[m + \frac{m(m+2)}{1-x^2}x^2 - \frac{2(m+1)mx^2}{1-x^2} + l(l+1) - m(m+1)\right]z_2 = 0$$

$$\{\ \} = l(l+1) + \frac{m^2+2m-2m^2-2m}{1-x^2}x^2 + m - m^2 - m$$

$$= l(l+1) - \frac{m^2x^2}{1-x^2} - m^2 = l(l+1) - \frac{m^2}{1-x^2}$$

BS K

$$(1-x^2) z_2'' - 2x z_2' + \left[\ell(\ell+1) - \frac{m^2}{1-x^2} \right] z_2 = 0$$

Sonuç: Birleşik Legendre fonk-u $P_\ell^m(x) = (1-x^2)^{m/2} \dfrac{d^m}{dx^m} P_\ell(x)$

birleşik Leg. denk.ini sağlar.

Rodriques Bağıntısı.

$$P_\ell^m(x) = \frac{1}{2^\ell \ell!} (1-x^2)^{m/2} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2-1)^\ell$$

BS K

3.9. Birleşik Legendre Fonk. larının Özellikleri.

Teorem 3.10. (i) $P_\ell^0(x) = P_\ell(x)$

(ii) $P_\ell^m(x) = 0$ eğer $m > \ell$ ise.

Teorem 3.11. Diklik Bağıntısı.

$$\boxed{\int_{-1}^{+1} P_\ell^m(x)\, P_{\ell'}^m(x)\, dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\,\delta_{\ell\ell'}}$$

$$P_\ell^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x)$$

$m > 0$ olsun.

$$\frac{d}{dx}\left[\frac{d^{m-1}}{dx^{m-1}}P_\ell\right]dx = dv$$

$$\int_{-1}^{+1}\{P_\ell^m(x)\}^2 dx = \int_{-1}^{+1}(1-x^2)^m\left\{\frac{d^m}{dx^m}P_\ell(x)\right\}\left\{\frac{d^m}{dx^m}P_\ell(x)\right\}dx$$

$$= \left[\left\{\frac{d^{m-1}}{dx^{m-1}}P_\ell(x)\right\}\left\{(1-x^2)^m\frac{d^m}{dx^m}P_\ell(x)\right\}\right]_{-1}^{+1}$$

$$- \int_{-1}^{+1}\left\{\frac{d^{m-1}}{dx^{m-1}}P_\ell(x)\right\}\frac{d}{dx}\left\{(1-x^2)^m\frac{d^m}{dx^m}P_\ell(x)\right\}dx$$

Bir. Leg. Diff. Denk.

$$(1-x^2)\frac{d^2 Z_1}{dx^2} - 2(m+1)x\frac{dZ_1}{dx} + \{\ell(\ell+1) - m(m+1)\}Z_1 = 0$$

$$Z_1 = \frac{d^m}{dx^m}Z \qquad m \to m-1$$

$$(1-x^2)\frac{d^{m+1}}{dx^{m+1}}P_\ell(x) - 2mx\frac{d^m}{dx^m}P_\ell(x) + \{\ell(\ell+1) - m(m-1)\}\frac{d^{m-1}}{dx^{m-1}}P_\ell(x) = 0$$

$(1-x^2)^{m-1}$ ile çarpalım.

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$$(1-x^2)^m \frac{d^{m+1}}{dx^{m+1}} P_\ell - 2mx\,(1-x^2)^{m-1} \frac{d^m}{dx^m} P_\ell + (\ell+m)(\ell-m+1)(1-x^2)^{m-1} \frac{d^{m-1}}{dx^{m-1}} P_\ell = 0$$

$$\frac{d}{dx}\left\{ (1-x^2)^m \frac{d^m}{dx^m} P_\ell \right\} = -(\ell+m)(\ell-m+1)(1-x^2)^{m-1} \frac{d^{m-1}}{dx^{m-1}} P_\ell$$

$$\int_{-1}^{+1} \{ P_\ell^m(x) \}^2 dx = (\ell+m)(\ell-m+1)$$

$$\times \int_{-1}^{+1} \left\{ \frac{d^{m-1}}{dx^{m-1}} P_\ell \right\} \left\{ \frac{d^{m-1}}{dx^{m-1}} P_\ell \right\} (1-x^2)^{m-1}$$

$$= (\ell+m)(\ell-m+1) \int_{-1}^{+1} (1-x^2)^{m-1} \left\{ \frac{d^{m-1}}{dx^{m-1}} P_\ell \right\}^2$$

$$= (\ell+m)(\ell-m+1) \int_{-1}^{+1} \{ P_\ell^{m-1}(x) \}^2 dx$$

Bu sonuç itere edilir ise,

$$= (\ell+m)(\ell-m+1)(\ell+m-1)(\ell-m+2)$$

$$\times \int_{-1}^{+1} \{ P_\ell^{m-1}(x) \}^2 dx$$

m defa iterasyon

$$\int_{-1}^{+1} [P_\ell^m(x)]^2 dx = (\ell+m)(\ell+m-1) \cdots (\ell+1)\,(\ell-m+1)(\ell-m+2) \cdots \ell$$

$$\times \int_{-1}^{+1} \{ P_\ell^0(x) \}^2 dx$$

$$= (\ell+m)(\ell+m-1)(\ell+1)\ell(\ell-1) \cdots (\ell-m+2)(\ell-m+1) \frac{2}{2\ell+1}$$

$$= \frac{(\ell+m)!}{(\ell-m)!} \frac{2}{2\ell+1}$$

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$m<0$ olsun.

$n>0$ olmak üzere $m=-n$ diyelim.

$$\int_{-1}^{+1}\left[P_\ell^m(x)\right]^2 dx=\int_{-1}^{+1}\left[P_\ell^{-n}(x)\right]^2 dx \qquad P_\ell^{-m}=(-1)^m\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m$$

$$=\left\{(-1)^n\frac{(\ell-n)!}{(\ell+n)!}\right\}^2\int_{-1}^{+1}\left[P_\ell^n(x)\right]^2 dx$$

$$=\left[\frac{(\ell-n)!}{(\ell+n)!}\right]^2\frac{(\ell+n)!}{(\ell-n)!}\int_{-1}^{+1}\left[P_\ell^0(x)\right]^2 dx$$

$$=\frac{(\ell-n)!}{(\ell+n)!}\frac{2}{2\ell+1}=\frac{(\ell+m)!}{(\ell-m)!}\frac{2}{2\ell+1}$$

Teorem. 3.12.) Türetme Bağıntıları:

(i) $P_\ell^{m+1}-\dfrac{2mx}{\sqrt{1-x^2}}P_\ell^m+\left[\ell(\ell+1)-m(m-1)\right]P_\ell^{m-1}=0$

(ii) $(2\ell+1)\,x\,P_\ell^m=(\ell+m)P_{\ell-1}^m+(\ell-m+1)P_{\ell+1}^m$

(iii) $\sqrt{1-x^2}\,P_\ell^m=\dfrac{1}{2\ell+1}\left[P_{\ell+1}^{m+1}-P_{\ell-1}^{m+1}\right]$

(iv) $\sqrt{1-x^2}\,P_\ell^m=\dfrac{1}{2\ell+1}\left[(\ell+m)(\ell+m-1)P_{\ell-1}^{m-1}-(\ell-m+1)(\ell-m+2)P_{\ell+1}^{m-1}\right]$

İspat:

(i) $\dfrac{d^m}{dx^m}P_\ell(x)=P_\ell^{(m)}$ diyelim

$P_\ell^m(x)=(1-x^2)^{m/2}P_\ell^{(m)}(x)$ olur.

$$(1-x^2)\frac{d^2 z_1}{dx^2} - 2(m+1)x\frac{dz_1}{dx} + [\ell(\ell+1) - m(m+1)]z_1 = 0$$

$Z = P_\ell(x)$ $\quad Z_1 = P_\ell^{(m)}(x)$ dır.

$$(1-x^2)\frac{d^2}{dx^2}P_\ell^{(m)} - 2(m+1)x\frac{d}{dx}P_\ell^{(m)} + [\ell(\ell+1) - m(m+1)]P_\ell^{(m)} = 0$$

$$(1-x^2)P_\ell^{(m+2)} - 2(m+1)xP_\ell^{(m+1)} + [\ell(\ell+1) - m(m+1)]P_\ell^{(m)} = 0$$

(*) $(1-x^2)^{m/2}$ ile çarpalım.

$$(1-x^2)^{\frac{m}{2}+1}P_\ell^{(m+2)} - 2(m+1)x(1-x^2)^{m/2}P_\ell^{(m+1)} + [\ell(\ell+1) - m(m+1)]P_\ell^{(m)}(1-x^2)^{m/2} = 0$$

$$P_\ell^{m+2}(x) - 2(m+1)x\frac{1}{\sqrt{1-x^2}}P_\ell^{m+1}(x) + [\ell(\ell+1) - m(m+1)]P_\ell^m(x) = 0$$

$m \to m-1$

$$P_\ell^{m+1}(x) - \frac{2mx}{\sqrt{1-x^2}}P_\ell^m(x) + [\ell(\ell+1) - m(m-1)]P_\ell^{m-1}(x) = 0$$

(ii) teorem 3.8 (ii) den

$$\frac{d^m}{dx^m}(uv) = \sum_{r=0}^{m}\binom{m}{r}\frac{d^r u}{dx^r}\frac{d^{m-r} v}{dx^{m-r}}$$

$$(\ell+1)P_{\ell+1} - (2\ell+1)xP_\ell + \ell P_{\ell-1} = 0$$

yok et!!

m defa türetelim.

$$(\ell+1)P_{\ell+m}^{(m)} - (2\ell+1)\{xP_\ell^{(m)} + mP_\ell^{(m-1)}\} + \ell P_{\ell-1}^{(m)} = 0$$

teorem 3.8. (iv) den

$$P_{\ell+1}^{(1)} - P_{\ell-1}^{(1)} = (2\ell+1)P_\ell \qquad (m-1) \text{ defa türet.}$$

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$$P_{\ell+1}^{(m)} - P_{\ell-1}^{(m)} = (2\ell+1)\, P_\ell^{(m-1)} \Longleftarrow (**)$$

P_ℓ^{m-1}'i yoketmek için bu denklemi kullanalım.

$$(\ell+1) P_{\ell+1}^{(m)} - (2\ell+1) x P_\ell^{(m)} - m\left\{ P_{\ell+1}^{(m)} - P_{\ell-1}^{(m)} \right\} + \ell P_{\ell-1}^{(m)} = 0$$

(*) $(1-x^2)^{m/2}$ ile çarpalım.

$$(\ell+1) P_{\ell+1}^{m} - (2\ell+1) x P_\ell^{m} + (\ell+m) P_{\ell-1}^{m} - m P_{\ell+1}^{m} = 0$$

$$(\ell+1-m) P_{\ell+1}^{m} - (2\ell+1) x P_\ell^{m} + (\ell+m) P_{\ell-1}^{m} = 0$$

(iii) $$P_{\ell+1}^{(m)} - P_{\ell-1}^{(m)} = (2\ell+1)\, P_\ell^{(m-1)} \Longleftarrow (**)$$

Bunu $(1-x^2)^{m/2}$ ile çarp.

$\cancel{(\ell+1) P_{\ell+1}^{m}}$

$$P_{\ell+1}^{m} - P_{\ell-1}^{m} = (2\ell+1)(1-x^2)^{1/2} P_\ell^{m-1} \quad (*) \longrightarrow$$

$m \to m+1$ koy!

$$P_{\ell+1}^{m+1} - P_{\ell-1}^{m+1} = (2\ell+1)(1-x^2)^{1/2} P_\ell^{m}$$

(iv) (ii) deki xP_ℓ^m 'yi (i)'ye yerleştir.

$$xP_\ell^m = \frac{1}{2\ell+1}\left\{(\ell+m)P_{\ell-1}^m + (\ell-m+1)P_{\ell+1}^m\right\}$$

$$P_\ell^{m+1} - \frac{2m}{\sqrt{1-x^2}}\frac{1}{2\ell+1}\left\{(\ell+m)P_{\ell-1}^m + (\ell-m+1)P_{\ell+1}^m\right\}$$
$$+\left\{\ell(\ell+1)-m(m+1)\right\}P_\ell^{m-1} = 0$$

(*) 'dan P_ℓ^{m-1} çek ve bu denk. de koy!

$$P_\ell^{m+1} - \frac{2m}{\sqrt{1-x^2}}\frac{1}{2\ell+1}\left\{(\ell+m)P_{\ell-1}^m + (\ell-m+1)P_{\ell+1}^m\right\}$$
$$+\left\{\ell(\ell+1)-m(m-1)\right\}\frac{1}{\sqrt{1-x^2}}\frac{1}{2\ell+1}\left\{P_{\ell+1}^m - P_{\ell-1}^m\right\} = 0$$

$$\Rightarrow \sqrt{1-x^2}\,P_\ell^{m+1} = \frac{1}{2\ell+1}\left\{(\ell+m)(\ell+m+1)P_{\ell-1}^m - (\ell-m)(\ell-m+1)P_{\ell+1}^m\right\}$$

$m \to m-1$ konulur ise teorem ispatlanır.

3.10. 2. türden Legendre Fonk.ları:

L. denk.nin çözümleri : $y_1(x)$ ve $y_2(x)$

$-1 \leq x \leq +1$ aralığında tam ℓ için çözümler $\begin{cases} y_1(x) & \text{çift } \ell \\ y_2(x) & \text{tek } \ell. \end{cases}$

Diğer seriler iki durumda da sonsuz kalırlar. $|x|>1$ için iki bağımsız çözümden bir tanesi $P_\ell(x)$, ikincisi $Q_\ell(x)$ olur.

Teorem 13) Leg. denk.nin 2. bağımsız çözümü

$$Q_\ell(x) = \frac{1}{2} P_\ell(x) \ln \frac{1+x}{1-x} - \sum_{r=0}^{[\ell/2]} \frac{(2\ell-4r-1)}{(2r+1)(\ell-r)} P_{\ell-2r-1}(x) \qquad \ell \geq 1$$

$$Q_0(x) = \frac{1}{2} \ln \frac{1+x}{1-x} \quad ; \quad \left[\frac{\ell-1}{2}\right] : \begin{cases} \frac{\ell-1}{2} & \ell : \text{tek} \\ \frac{\ell-2}{2} & \ell : \text{çift.} \end{cases}$$

Q_ℓ P_ℓ'den bağımsızdır ve 2. tip Leg. fonk.u adını alır.

$Q_\ell(x)$ ıraksaktır.

$$Q_1(x) = \frac{x}{2} \ln \frac{1+x}{1-x} - 1$$

$$Q_2(x) = \frac{1}{4}(3x^2-1) \ln \frac{1+x}{1-x} - \frac{3}{2}x$$

$$Q_3(x) = \frac{1}{4}(5x^3-3x) \ln \frac{1+x}{1-x} - \frac{5}{2}x^2 + \frac{2}{3}$$

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Teorem 3.14.

$$\frac{1}{x-y} = \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(x) Q_\ell(y) \qquad x>1 \ , |y| \leq 1$$

Teorem. 3.15.

$$Q_\ell(x) = \frac{1}{2} \int_{-1}^{+1} \frac{P_\ell(y)}{x-y} \, dy \qquad \text{Neumann formülü}$$

Teorem 3.16. Teorem 3.8 [(ii) → (ix) olan sonuçlar] $P_\ell \to Q_\ell$ alınmağında doğrudur.

Teorem. 3.17.

$$Q_\ell^m = (1-x^2)^{m/2} \frac{d^m}{dx^m} Q_\ell(x)$$

2. tür birleşik Legendre fonk.

3.11. Küresel Harmonikler :

$$\nabla^2 \Psi = 0$$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Psi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\varphi^2} = 0$$

$$\Psi = \frac{U(r)}{r}\Theta(\theta)\Phi(\varphi)$$

$$\Theta\Phi\frac{d^2U}{dr^2} + \frac{U\Phi}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{U\Theta}{r^2\sin^2\theta}\frac{d^2\Phi}{d\varphi^2} = 0$$

$$\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{r^2}\left\{\underbrace{\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\varphi^2}}_{-\ell(\ell+1)}\right\} = 0$$

$$\frac{d^2U}{dr^2} - \frac{\ell(\ell+1)}{r^2}U = 0$$

$$\frac{1}{\Theta\sin\theta}\left\{\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta}{d\theta}\right] + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\varphi^2}\right\} = -\ell(\ell+1)$$

$$\frac{\sin\theta}{\Theta}\left[\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right] + \ell(\ell+1)\sin^2\theta + \underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}}_{-m^2} = 0$$

$$\frac{d^2\Phi}{d\varphi^2} = -m^2\Phi$$

BS K

1.

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left\{\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right\}\Theta=0$$

$0\leq\theta\leq\pi \qquad 0\leq\varphi\leq 2\pi.$

$$\Phi=Ae^{im\varphi}+Be^{-im\varphi}$$

Çözümün sürekli olmasından $\Phi(2\pi)=\Phi(0)$ m tamsayı ve pozitif.

$$\cos\theta=x \Rightarrow \frac{1}{\sin\theta}\frac{d}{d\theta}=-\frac{d}{dx}$$

$$\sin\theta\frac{d}{d\theta}=\sin^2\theta\left(-\frac{d}{dx}\right)=-(1-x^2)\frac{d}{dx}$$

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right]+\left[(\ell)(\ell+1)-\frac{m^2}{1-x^2}\right]\Theta=0$$

$\theta=0$ ve π $(x=\pm1)$

$$\Theta=P_\ell^m(x)=P_\ell^m(\cos\theta)$$

$$\Theta\Phi=\left(Ae^{im\varphi}+Be^{-im\varphi}\right)P_\ell^m(\cos\theta)$$

$$=A_1e^{im\varphi}P_\ell^m+A_2e^{-im\varphi}P_\ell^{-m}$$

$$A_1=A \qquad A_2=(-1)^m\frac{(l+m)!}{(l-m)!}B.$$

$$y_\ell^m(\theta,\varphi) = e^{im\varphi} P_\ell^m(\cos\theta) \quad \text{diyelim.}$$

Genel çözüm.

$$\Psi = \frac{u}{r}\,\Theta\,\Phi = \frac{u}{r}\left\{A_1 y_\ell^m(\theta,\varphi) + A_2 y_\ell^{-m}(\theta,\varphi)\right\}$$

Tanım : $\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \left\{Y_\ell^m(\theta,\varphi)\right\}\left\{Y_{\ell'}^{m'}(\theta,\varphi')\right\} = \delta_{\ell\ell'}\delta_{mm'}$

olacak şekilde $Y_\ell^m(\theta,\varphi)$ tanımlayalım.

$$Y_\ell^m = (-1)^m \frac{1}{\sqrt{2\pi}} \sqrt{\frac{(2\ell+1)(\ell-m)!}{2(\ell+m)!}}\, y_\ell^m(\theta,\varphi)$$

$$= (-1)^m \frac{1}{\sqrt{2\pi}} \sqrt{\frac{(2\ell+1)(\ell-m)!}{2(\ell+m)!}}\, e^{im\varphi} P_\ell^m(\cos\theta)$$

Ortonormal özelliği hemen gösterilebilir.

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \left\{Y_\ell^m\right\}\left\{Y_{\ell'}^{m'}\right\} = (-1)^{m+m'} \frac{1}{2\pi} \sqrt{\frac{(2\ell+1)(2\ell'+1)(\ell-m)!(\ell'-m')!}{4(\ell+m)!(\ell'+m')!}}$$

$$\times \underbrace{\int_0^{2\pi} e^{i(m-m')\varphi} d\varphi}_{2\pi\delta_{mm'}} \underbrace{\int_0^{\pi} P_\ell^m(\cos\theta) P_{\ell'}^{m'}(\cos\theta) \sin\theta\, d\theta}_{\frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell\ell'}}$$

$$= \delta_{\ell\ell'}\delta_{mm'}$$

Teorem 3.18.

$$\{Y_\ell^m(\theta,\varphi)\}^* = (-1)^m Y_\ell^{-m}(\theta,\varphi)$$

İspat:

$$\{Y_\ell^m(\theta,\varphi)\}^* = (-1)^m \frac{1}{\sqrt{2\pi}}\left(\frac{(2\ell+1)}{2}\frac{(\ell-m)!}{(\ell+m)!}\right)^{1/2} e^{-im\varphi} P_\ell^m(\cos\theta)$$

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)$$

$$= (-1)^m \frac{1}{\sqrt{2\pi}}\left(\frac{(2\ell+1)}{2}\frac{(\ell-m)!}{(\ell+m)!}\right)^{1/2} e^{-im\varphi}(-1)^{-m}\frac{(\ell+m)!}{(\ell-m)!} P_\ell^{-m}(x)$$

$$= (-1)^m Y_\ell^{-m}(\theta,\varphi)$$

$$Y_0^0 = \sqrt{\frac{1}{4\pi}}$$

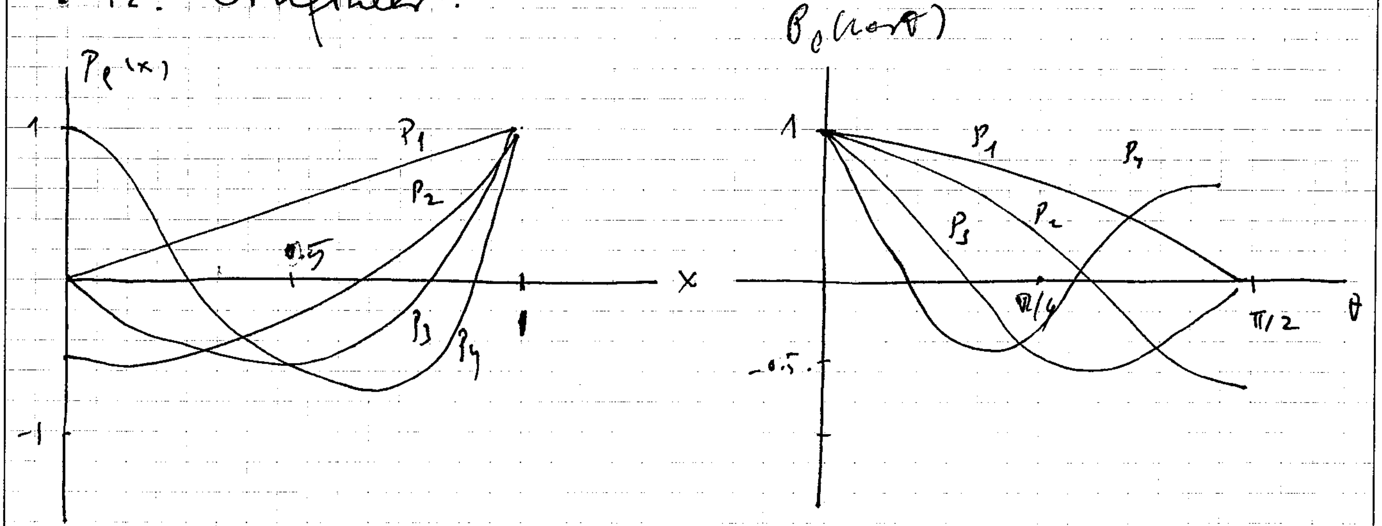
$$Y_1^{\pm 1} = \pm\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\varphi} \qquad Y_2^{\pm 1} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\varphi}$$

$$Y_2^0 = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right)$$

BS K

8.12. Grafikler.



BS K

$$W(x,t) = (1-2xt+t^2)^{-1/2} = \sum_n t^n P_n$$

$$\frac{\partial w}{\partial t} = \left(-\frac{1}{2}\right)(-2x+2t)(1-2xt+t^2)^{-3/2}$$

$$(1-2xt+t^2)\frac{\partial w}{\partial t} + (t-x)W = 0$$

$$(1-2xt+t^2)\sum_{n=0} n t^{n-1} P_n(x) + (t-x)\sum_{n=0}^{\infty} t^n P_n(x) = 0$$

$$\sum_{n=0}^{\infty} n t^{n-1} P_n - 2x\sum_{n=0} n t^n P_n + \sum_{n=0} n t^{n+1} P_n + \sum_{n=0} t^{n+1} P_n - x\sum_{n=0} t^n P_n = 0$$

$$\sum_{n=0} (n+1) t^n P_{n+1} - 2x\sum_{n=0} n t^n P_n - x\sum_{n=0} t^n P_n + \sum_{n=0}^{\infty} \underbrace{(n+1) t^{n+1} P_n}_{n \to n-1 \,:\, \sum_{n=1}^{\infty} n P_{n-1}} = 0$$

(1) $$\boxed{(n+1)P_{n+1} - (2n+1)xP_n + nP_{n-1} = 0}$$

t^n'nin katsayısı.

$$\frac{\partial w}{\partial x} = \left(-\frac{1}{2}\right)(-2t)(1-2xt+t^2)^{-3/2}$$

$$(1-2xt+t^2)\frac{\partial w}{\partial x} - tW = 0$$

$$(1-2xt+t^2)\sum_{n=0} P_n'(x)\, t^n - \sum_{n=0}^{\infty} P_n(x)\, t^{n+1} = 0$$

$$\sum_{n=0} P_n' t^n - 2x\sum_{n=0} P_n' t^{n+1} + \sum_{n=0} P_n' t^{n+2} - \sum_{n=0} P_n t^{n+1} = 0$$

(2) $$\boxed{P_{n+1}' - 2xP_n' + P_{n-1}' - P_n = 0}$$

BS K

1 denk.ini türetip,

$$\cancel{(n+1)} P_{n+1}' - (2n+1) P_n - \cancel{(2n+1)} x P_n' + n P_{n-1}' = 0$$

bu değil.

$$\cancel{P_{n+1}'} - 2x P_n' + P_{n-1}' - P_n' = 0 \quad \text{(2) idi.} \quad / n \text{ 'le çarp.}$$

---

3 $\cancel{P_{n+1}' - 2n P_n - x P_n' = 0}$ $\boxed{P_{n+1}' - x P_n' = (n+1) P_n}$ $n=0,1,2,\ldots$

P_{n+1}' yok edilse idi.

4 $\boxed{x P_n' - P_{n-1}' = n P_n}$ $n=1,2,3,\ldots$

3 de $n \to n-1$ al.

$$P_n' - x P_{n-1}' = n P_{n-1}$$

$$x P_n' - P_{n-1}' = n P_n \quad (4).$$

P_{n-1}' 'leri yok et.

$$\boxed{(x^2-1) P_n' = n x P_n - n P_{n-1}}$$

BS K

Örnek 1. $\displaystyle\int_{-1}^{+1} x^2 P_{\ell+1}(x) P_{\ell-1}(x)\, dx = \frac{2\ell(\ell+1)}{(4\ell^2-1)(2\ell+3)}$

$$I = \int_{-1}^{+1} \{x P_{\ell+1}(x)\}\{x P_{\ell-1}(x)\}\, dx = ?$$

$$x P_\ell(x) = \frac{\ell+1}{2\ell+1} P_{\ell+1}(x) + \frac{\ell}{2\ell+1} P_{\ell-1}(x) \qquad (ii)$$

$$\Rightarrow I = \int_{-1}^{+1} \left\{ \frac{\ell+2}{2\ell+3} P_{\ell+2} + \frac{\ell+1}{2\ell+3} P_\ell \right\}\left\{ \frac{\ell}{2\ell-1} P_\ell + \frac{\ell-1}{2\ell-1} P_{\ell-2} \right\} dx$$

$$= \frac{\ell(\ell+1)}{(2\ell+3)(2\ell-1)} \int_{-1}^{+1} P_\ell P_\ell \, dx = \frac{\ell(\ell+1)\, 2}{(2\ell+3)(2\ell-1)(2\ell+1)}$$

Sonuç: P_ℓ ℓ. dereceden bir polinom.

integral: $2 + (\ell+1) + (\ell-1) = 2(\ell+1)$. dereceden polinom. çift.

$$\int_{-1}^{+1} \ldots = 2 \int_0^1 x^2 P_{\ell+1} P_{\ell-1}\, dx.$$

Örnek 2. ℓ tek olduğunda $\int_0^1 P_\ell(x)\, dx$ hesaplayınız.

$$P'_{\ell+1}(x) - P'_{\ell-1}(x) = (2\ell+1) P_\ell(x) \qquad (iv)$$

$$\int_0^1 P_\ell(x)\, dx = \frac{1}{2\ell+1} \int_0^1 \{ P'_{\ell+1} - P'_{\ell-1} \}\, dx$$

$$= \frac{1}{2\ell+1} \Big[P_{\ell+1}(x) - P_{\ell-1}(x) \Big]_0^1$$

$$= \frac{1}{2\ell+1} \{ P_{\ell+1}(1) - P_{\ell-1}(1) - P_{\ell+1}(0) + P_{\ell-1}(0) \}$$

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$$P_{2\ell}(0) = (-1)^{\ell} \frac{(2\ell)!}{2^{2\ell}(\ell!)^2}$$

$$\int_0^1 P_\ell(x)dx = \frac{1}{2\ell+1}\left\{\cancel{1} - \cancel{1} - \frac{(-1)^{\frac{\ell+1}{2}}(\ell+1)!}{2^{\ell+1}\left[\left(\frac{\ell+1}{2}\right)!\right]^2} + \frac{(-1)^{\frac{\ell-1}{2}}(\ell-1)!}{2^{\ell-1}\left[\left(\frac{\ell-1}{2}\right)!\right]^2}\right\}$$

$$= \frac{1}{2\ell+1}\frac{(-1)^{\frac{\ell-1}{2}}(\ell-1)!}{\left[\left(\frac{\ell-1}{2}\right)!\right]^2}\frac{1}{2^{\ell-1}}\left[1 - \frac{(-1)\,\ell(\ell+1)}{2^2\left(\frac{\ell+1}{2}\right)^2}\right]$$

$$= \frac{1}{2\ell+1}\frac{(-1)^{\frac{\ell-1}{2}}}{2^{\ell-1}}\frac{(\ell-1)!}{\left[\left(\frac{\ell-1}{2}\right)!\right]^2}\left[1 + \frac{\ell}{\ell+1}\right]$$

$$\int_0^1 P_\ell(x)\,dx = \frac{(-1)^{(\ell-1)/2}(\ell-1)!}{2^\ell\left(\frac{\ell}{2}+\frac{1}{2}\right)\left(\frac{\ell}{2}-\frac{1}{2}\right)!\left(\frac{\ell}{2}-\frac{1}{2}\right)!} = \frac{(-1)^{(\ell-1)/2}(\ell-1)!}{2^\ell\left(\frac{\ell}{2}+\frac{1}{2}\right)!\left(\frac{\ell}{2}-\frac{1}{2}\right)!}$$

Örnek. 3. $f(x)=|x|$ $\quad -1 \le x \le +1$ ise $f(x)$'i $\sum C_r P_r$ serisine açınız.

$$C_r = \left(r+\frac{1}{2}\right)\int_{-1}^{+1}|x|P_r(x)dx$$

$|x|$: çift

$P_r(x)$: $\begin{cases} \text{çift} & r \text{ çift} \\ \text{tek} & r \text{ tek}\end{cases}$ $\Rightarrow$ $\begin{cases} r \text{ tek} \Rightarrow |x|P_r(x) \text{ tek} \\ r \text{ çift} \Rightarrow |x|P_r(x) \text{ çift}\end{cases}$

ve $\int_{-1}^{+1}|x|P_r(x)=0$.

r çift ise

$$\int_{-1}^{+1}|x|P_r(x)\,dx = 2\int_0^1|x|P_r(x)dx = 2\int_0^1 xP_r(x)dx$$

$$C_r = (2r+1)\int_0^1 xP_r(x)dx$$

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$$C_r = \cancel{(2r+1)} \int_0^1 \left\{ \frac{r+1}{\cancel{2r+1}} P_{r+1}(x) + \frac{r}{\cancel{2r+1}} P_{r-1}(x) \right\} dx$$

$$= \int_0^1 \left[(r+1) P_{r+1} + r P_{r-1} \right] dx$$

r çift ise $(r+1)$ ve $(r-1)$ tektir. 2. problemin sonucu kullanılır.

$$C_r = (r+1) \frac{(-1)^{r/2} r!}{2^{r+1} \left(\frac{r}{2}+1\right)! \left(\frac{r}{2}\right)!} + r \frac{(-1)^{\frac{r}{2}-1} (r-2)!}{2^{r-1} \left(\frac{r}{2}\right)! \left(\frac{r}{2}-1\right)!}$$

$$= \frac{(-1)^{r/2} (r-2)!}{2^{r-1} \left(\frac{r}{2}\right) \left(\frac{r}{2}-1\right)!} \left\{ \underbrace{\frac{(r+1) r (r-1)}{2^2 \left(\frac{r}{2}+1\right)\left(\frac{r}{2}\right)} - r}_{\frac{(r+1)(r-1)}{r+2} - r \,=\, \frac{\cancel{r^2} - 1 - \cancel{r^2} - 2r}{2\left(\frac{r}{2}+1\right)}} \right\}$$

$$C_r = \frac{(-1)^{\frac{r}{2}+1} (2r+1)(r-2)!}{2^r \left(\frac{r}{2}+1\right)! \left(\frac{r}{2}-1\right)!}$$

$$f(x) = \sum_{n=0} \frac{(-1)^{n+1} (4n+1)(2n-2)!}{2^{2n} (n+1)! (n-1)!} P_{2n}(x)$$

$$f(x) = \sum_{r=0}^{\infty} C_r P_r(x) = \sum_{r=0}^{\infty} C_{2r} P_{2r}(x) + \cancel{\sum_{r=0}^{\infty} C_{2r+1} P_{2r+1}(x)}$$

Örnek 4. $x>1 \Rightarrow P_\ell(x) < P_{\ell+1}(x)$ olduğunu gösteriniz.

$P_{\ell-1}(x) < P_\ell(x)$ olsun.

$x>1 \Rightarrow P_\ell(x)>0$ olduğunu Rodrigues formülünden.

3.8.(iii) 'den: $(\ell+1)\dfrac{P_{\ell+1}}{P_\ell} - (2\ell+1)x + \dfrac{\ell P_{\ell-1}}{P_\ell} = 0$

$$\frac{P_{\ell+1}}{P_\ell} = \frac{(2\ell+1)}{(\ell+1)}x - \frac{\ell}{\ell+1}\,\frac{P_{\ell-1}}{P_\ell}$$

$x>1$ ve $\dfrac{P_{\ell-1}}{P_\ell}<1$ olduğundan

$$\frac{P_{\ell+1}}{P_\ell} > \frac{(2\ell+1)}{\ell+1} - \frac{\ell}{\ell+1} = \frac{\ell+1}{\ell+1} = 1$$

$$P_{\ell+1} > P_\ell .$$

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Örnek. 5. $\ell\{Q_\ell P_{\ell-1} - Q_{\ell-1}P_\ell\} = (\ell-1)\{Q_{\ell-1}P_{\ell-2} - Q_{\ell-2}P_{\ell-2}\}$ olduğunu ispatla ve buradan $\ell\{Q_\ell P_{\ell-1} - Q_{\ell-1}P_\ell\} = -1$ olduğunu göster.

$$(\ell+1)P_{\ell+1} - (2\ell+1)xP_\ell + \ell P_{\ell-1} = 0$$

$\ell \to \ell-1$:

$$\ell P_\ell - (2\ell-1)xP_{\ell-1} + (\ell-1)P_{\ell-2} = 0 \qquad / Q_{\ell-1}$$

$$\ell Q_\ell - (2\ell-1)xQ_{\ell-1} + (\ell-1)Q_{\ell-2} = 0 \qquad / P_{\ell-1}$$

$-$

$$\ell\{P_\ell Q_{\ell-1} - Q_\ell P_{\ell-1}\} + (\ell-1)\{P_{\ell-2}Q_{\ell-1} - Q_{\ell-2}P_{\ell-1}\} = 0$$

$$\underbrace{\ell\{Q_\ell P_{\ell-1} - Q_{\ell-1}P_\ell\}}_{F(\ell)\text{ olsun.}} = (\ell-1)\{Q_{\ell-1}P_{\ell-2} - Q_{\ell-2}P_{\ell-1}\}$$

$$F(\ell) = F(\ell-1)$$

$$F(\ell) = F(\ell-1) = F(\ell-2) = \ldots = F(1)$$

$$F(\ell) = F(1)$$

$$\ell\{Q_\ell P_{\ell-1} - Q_{\ell-1}P_\ell\} = Q_1P_0 - Q_0P_1$$

$P_0 = 1\,;\ P_1 = x$

$Q_0 = \frac{1}{2}\ln\frac{1+x}{1-x}$

$Q_1 = \frac{1}{2}x\ln\frac{1+x}{1-x} - 1$

$$= \cancel{\frac{1}{2}x\ln\frac{1+x}{1-x}} - 1 - \cancel{\frac{1}{2}x\ln\frac{1+x}{1-x}}$$

$$= -1$$

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prb. 1.) $\int_{-1}^{+1} x P_\ell(x) P_{\ell-1}(x) dx = \frac{2\ell}{4\ell^2-1} = I$

$$xP_\ell = \frac{\ell+1}{2\ell+1} P_{\ell+1} + \frac{\ell}{2\ell+1} P_{\ell-1}$$

$$\Rightarrow I = \int_{-1}^{+1} \left\{ \frac{\ell+1}{2\ell+1} P_{\ell+1} + \frac{\ell}{2\ell+1} P_{\ell-1} \right\} P_{\ell-1}(x) dx$$

$$= \frac{\ell}{2\ell+1} \int_{-1}^{+1} P_{\ell-1} P_{\ell-1} dx = \frac{2\ell}{4\ell^2-1}$$

$$\frac{2}{2(\ell-1)+1} \delta_{\ell-1\,\ell-1} \quad 1$$

$$\left(\frac{2}{2\ell-1}\right)$$

prb. 2.) $\int_{-1}^{+1} (1-x^2) P_\ell'(x) P_m'(x) dx = \frac{2\ell(\ell+1)}{2\ell+1} \delta_{\ell m} = I$

$$(x^2-1) P_\ell' = \ell x P_\ell - \ell P_{\ell-1}$$

$$I = \int_{-1}^{+1} [\ell P_{\ell-1} - \ell x P_\ell] P_m' dx$$

$$xP_\ell = \frac{\ell+1}{2\ell+1} P_{\ell+1} + \frac{\ell}{2\ell+1} P_{\ell-1}$$

$$I = \int_{-1}^{+1} \left\{ \ell P_{\ell-1} - \frac{\ell(\ell+1)}{2\ell+1} P_{\ell+1} - \frac{\ell^2}{2\ell+1} P_{\ell-1} \right\} P_m' dx$$

$$(\cancel{2\ell^2} + \ell - \cancel{\ell^2}) = \ell(\ell+1)$$

$$= \frac{\ell(\ell+1)}{2\ell+1} \int_{-1}^{+1} [P_{\ell-1} - P_{\ell+1}] \underbrace{P_m' dx}_{dv}$$

$$= \frac{\ell(\ell+1)}{2\ell+1} \Big\{ \underbrace{[P_{\ell-1} - P_{\ell+1}] P_m \Big|_{-1}^{+1}}_{(-1)^{\ell-1} - (-1)^{\ell+1} = 0} + \int_{-1}^{+1} \underbrace{[P_{\ell+1}' - P_{\ell-1}']}_{(2\ell+1) P_\ell(x)} P_m dx$$

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$$I = \frac{2\ell(\ell+1)}{2\ell+1}\,\delta_{\ell m}.$$

prb. 6.) $u_n = \int_{-1}^{+1} x^{-1} P_n(x) P_{n-1}(x)\,dx$ ise

$n u_n + (n-1) u_{n-1} = 2$ olduğunu göster ve u_n'yi hesapla.

$$x P_\ell = \frac{\ell+1}{2\ell+1} P_{\ell+1} + \frac{\ell}{2\ell+1} P_{\ell-1}$$

$$\frac{P_{n-1}}{x} = \frac{2n+1}{n} P_n - \frac{n+1}{n} \frac{P_{n+1}}{x}$$

$$u_n = \int_{-1}^{+1} P_n(x) \left\{ \frac{2n+1}{n} P_n - \frac{n+1}{n} \frac{P_{n+1}}{x} \right\} dx$$

$$= \frac{2n+1}{n} \int_{-1}^{+1} P_n^2\,dx - \frac{n+1}{n} \int \frac{P_n}{x} P_{n+1}\,dx$$

$$= \frac{\cancel{2n+1}}{n} \frac{2}{\cancel{2n+1}} - \frac{n+1}{n} u_{n+1}$$

$n \to n-1$

$$n u_n + (n+1) u_{n+1} = 2 \Rightarrow n u_n + (n-1) u_{n-1} = 2$$

$$u_1 = \int_{-1}^{+1} \frac{P_1 P_0}{x}\,dx = 2$$

$$2u_2 = 2 - 1.u_1 = 0$$

$$3u_3 = 2 - 2u_2 = 2 \Rightarrow u_3 = 2/3$$

$$4u_4 = 2 - 3u_3 = 0$$

$$5u_5 = 2 - 4u_4 = 2 \Rightarrow u_5 = 2/5$$

$$u_n = \begin{cases} 2/n & n : \text{tek} \\ 0 & n : \text{çift.} \end{cases}$$

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