BÖLÜM 5

HERMITE POLİNOMLARI

5.1. Hermite Denk.'i ve çözümü:

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0$$

uygulamalarda x'in tüm sonlu değerleri için sonlu çözümler ararız. ($x \to \infty$ iken $\exp(x^2/2)y(x) \to 0$)

$$z(x,s) = \sum_{r=0}^{\infty} a_r x^{r+s}$$

$$\frac{a_{r+2}}{a_r} = \frac{2(r-n)}{(r+1)(r+2)}$$

$s=0,1$

$s=0 \Rightarrow a_1$: belirsiz.

$s=0 \to$ iki bağımsız çözüm.

($x \to \infty$ iken iki çözüm de $e^{x^2/2}$ şeklinde davranır)

Bu da n negatif olmayan bir tamsayı ise olur.

$$a_r = -\frac{(r+1)(r+2)}{2(n-r)} a_{r+2}$$

$\to$ a_{n+2} ve bundan sonraki terimler sıfır olur.

$r \to n-2r$

$$z(x,0) = \sum a_{n-2r} x^{n-2r}$$

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$$r=1 \Rightarrow a_{n-2} = -\frac{(n-1)\,n}{2.2}\,a_n$$

$$r=2 \Rightarrow a_{n-4} = -\frac{(n-3)(n-2)}{2.4}\,a_{n-2}$$

$$= +\frac{(n-3)(n-2)(n-1)\,n}{2^2\,.\,2.4}\,a_n$$

$$\vdots$$

$$y = a_n\left\{x^n - \frac{n(n-1)}{2.2}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2^2\,.\,2.4}x^{n-4} + \dots\right.$$

$$\left. + (-1)^r\frac{n(n-1)\dots(n-2r+1)}{2^r\,2.4\dots 2r}x^{n-2r} + \dots\right\}$$

$$= a_n\sum_{r=0}^{[n/2]}(-1)^r\frac{n(n-1)\dots(n-2r+1)}{2^r\,.2.4\dots 2r}x^{n-2r}$$

$$[n/2] = \begin{cases} n/2 & n \text{ çift.} \\ (n-1)/2 & n\text{ : tek.} \end{cases}$$

$$= a_n\sum(-1)^r\frac{n!}{2^{2r}\,r!\,(n-2r)!}x^{n-2r}$$

$$H_n(x) = \sum_{0}^{[n/2]}(-1)^r\frac{n!}{r!\,(n-2r)!}(2x)^{n-2r}$$

$a_n = 2^n$ seçildi

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5.2.) Doğurucu Fonk.

Thm 5.1:

$$e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$$

İspat:

$$e^{2tx - x^2} = e^{2tx} \, e^{-x^2}$$

$$= \sum_{r=0}^{\infty} \frac{(2tx)^r}{r!} \sum_{s=0}^{\infty} \frac{(-t^2)^s}{s!}$$

$$= \sum_{r,s=0}^{\infty} (-1)^s \frac{(2x)^r}{r!\, s!} t^{r+2s}$$

$$r + 2s = n \qquad r = n - 2s$$

$$t^n : \quad \sum_{s=0}^{[n/2]} (-1)^s \frac{(2x)^{n-2s}}{(n-2s)!\, s!} = \frac{H_n(x)}{n!}$$

$$n - 2s \geq 0 \qquad s \leq n/2$$

$$s = \begin{cases} 0 \to n/2 & n: \text{çift} \\ 0 \to (n-1)/2 & n: \text{tek} \end{cases}$$

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5.3. Hermite Pol. için Diğer ifadeler

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

İspat:

$$F(t) = \sum_{n=0}^{\infty} \left(\frac{d^n F}{dt^n}\right)_{t=0} \frac{t^n}{n!} \quad \text{Taylor teor.}$$

$$e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$$

$$\Rightarrow \quad H_n(x) = \left[\frac{\partial^n}{\partial t^n} e^{2tx - t^2}\right]_{t=0}$$

$$= \left[\frac{\partial^n}{\partial t^n} e^{x^2 - (x-t)^2}\right]_{t=0}$$

$$= e^{x^2} \left\{\frac{\partial^n}{\partial t^n} e^{-(x-t)^2}\right\}_0$$

$$\frac{\partial}{\partial t} f(x-t) = -\frac{\partial}{\partial x} f(x-t)$$

$$\frac{\partial^n}{\partial t^n} f(x-t) = (-1)^n \frac{\partial^n}{\partial x^n} f(x-t)$$

$$\Rightarrow H_n(x) = (-1)^n e^{x^2} \left\{\frac{\partial^n}{\partial x^n} e^{-(x-t)^2}\right\}_{t=0}$$

$$= (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

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Thr. 5.3.

$$H_n(x) = 2^n \, e^{-\frac{1}{4}\frac{d^2}{dx^2}} \, x^n$$

İspat :

$$\frac{1}{2}\frac{d}{dx} e^{2tx} = t\, e^{2tx}$$

$$\Rightarrow \left(\frac{1}{2}\frac{d}{dx}\right)^n e^{2tx} = t^n e^{2tx}$$

$$e^{-\frac{1}{4}\frac{d^2}{dx^2}} e^{2tx} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)^n e^{2tx}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left(\frac{1}{2}\frac{d^{2n}}{dx^{2n}}\right) e^{2tx}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n} e^{2tx}$$

$$= e^{-t^2} e^{2tx} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$$

$$\Rightarrow e^{-\frac{1}{4}\frac{d^2}{dx^2}} \sum_{n=0}^{\infty} \frac{2^n x^n t^n}{n!} =$$

$$\Rightarrow e^{-\frac{1}{4}\frac{d^2}{dx^2}} \frac{2^n x^n}{n!} = \frac{H_n}{n!}$$

5.4. Hermite Pol.'nın Özel Değerleri ve Özel İfadeler.

$H_0(x) = 1$

$H_1(x) = 2x$

$H_2(x) = 4x^2 - 2$

$H_3(x) = 8x^3 - 12x$

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Thr. 5.4.

$$H_{2n}(0) = (-1)^n \frac{(2n)!}{n!} \qquad H_{2n+1}(0) = 0$$

İspat:

$$e^{-t^2} = \sum_{n=0}^{\infty} H_n(0) \frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!} = \sum_{n=0}^{\infty} H_n(0) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} H_{2n+1}(0) \frac{t^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} H_{2n}(0) \frac{t^{2n}}{(2n)!}$$

5.5. Hermite Polinomlarının Ortogonalliği

Thr. 5.5:

$$\int_{-\infty}^{+\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 2^n n! \sqrt{\pi}\, \delta_{nm}$$

İspat:

$$e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

$$e^{-s^2+2sx} = \sum_{m=0}^{\infty} H_m(x) \frac{s^m}{m!}$$

$$\int_{-\infty}^{+\infty} e^{-x^2} e^{-t^2+2tx} e^{-s^2+2sx}\, dx \quad \text{açılımında}$$

$t^n s^m / n!\, m!$ 'in katsayısı $\int_{-\infty}^{+\infty} e^{-x^2} H_n H_m\, dx$.

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$$\int_{-\infty}^{+\infty} e^{-x^2}\, e^{-t^2-2tx}\, e^{-s^2+2sx}\, dx$$

$$= e^{-t^2-s^2} \int_{-\infty}^{+\infty} \exp\{-[x^2-2(s+t)x]\}\, dx$$

$$= e^{-t^2-s^2} \int \exp\{-[x-(s+t)]^2+(s+t)^2\}\, dx$$

$$= e^{2st} \int \exp\{-[x-(s+t)]^2\}\, dx$$

$$= e^{2st} \int_{-\infty}^{+\infty} e^{-u^2}\, du = e^{2st}\sqrt{\pi}$$

$$= \sum_{n=0}^{\infty} \sqrt{\pi}\, \frac{2^n s^n t^n}{n!}$$

$$\frac{t^n s^m}{n!\,m!}\text{'in katsayısı} = \begin{cases} 0 & m \neq n \\ \sqrt{\pi}\, 2^n n! & m = n. \end{cases}$$

5.6. İndirgeme Bağıntıları:

Thr. 5.6

(i) $H_n' = 2n H_{n-1} \quad n \geq 1 \quad ; \quad H_0' = 0$

(ii) $H_{n+1} = 2x H_n - 2n H_{n-1} \quad n \geq 1 \quad ; \quad H_1(x) = 2x H_0(x)$

İspat:

(i) $$\frac{d}{dx}\left\{ e^{2tx - t^2} \right\} = \sum_{n=0}^{\infty} H_n'(x) \frac{t^n}{n!}$$

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$$\sum_0^\infty H_n' \frac{t^n}{n!} = 2t\, e^{2tx-t^2}$$

$$= 2t \sum_0^\infty H_n \frac{t^n}{n!}$$

$$= 2 \sum_0^\infty H_n \frac{t^{n+1}}{n!} = 2 \sum_{n=1}^\infty H_{n-1}(x) \frac{t^n}{(n-1)!}$$

t^n 'nin katsayıları $\begin{cases} 0 = H_n'(x) & n=0 \\ \dfrac{H_n'}{n!} = \dfrac{2H_{n-1}}{(n-1)!} & n \geq 1 \end{cases}$

$$H_n'(x) = 2n\, H_{n-1}(x).$$

(ii) t'ye göre türetilerek bulunur.

$$\frac{d}{dt}\left\{e^{2tx-t^2}\right\} = \sum_0^\infty \frac{H_n}{n!} n\, t^{n-1}$$

$$(2x-2t)\, e^{2tx-t^2} = \sum_0^\infty \frac{H_n}{n!} n\, t^{n-1}$$

$$2x \sum_0^\infty \frac{t^n}{n!} H_n - 2\sum_0^\infty \frac{t^{n+1}}{n!} H_n = \sum_{n=1}^\infty \frac{H_n}{\cancel{n!}\,(n-1)!} \cancel{n}\, t^{n-1}$$

$$n \to n-1 \qquad\qquad n \to n+1$$

~~$2xH_n$~~

$$2x \sum_0 \frac{t^n}{n!} H_n - 2\sum_1^\infty \frac{t^n}{(n-1)!} H_{n-1} = \sum_{n=0}^\infty \frac{t^n}{n!} H_{n+1}$$

$$2x \sum_0 \frac{t^n}{n!} H_n - 2\sum_0^\infty \frac{n\,t^n}{n!} H_{n-1} = \sum_0^\infty \frac{t^n}{n!} H_{n+1}$$

$$H_{n+1} = 2x H_n - 2n H_{n-1}$$

5.8. ÖRNEKLER

1°) $m<n$ ise $\dfrac{d^m}{dx^m}\{H_n(x)\} = \dfrac{2^m n!}{(n-m)!} H_{n-m}(x)$

olduğunu ispat ediniz.

Doğuran fonk. $e^{2tx-t^2} = \sum_{n=0}^{\infty} \dfrac{t^n}{n!} H_n(x)$

$$\frac{d^m}{dx^m} e^{2tx-t^2} = (2t)^m e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{d^m}{dx^m} H_n(x)$$

$$\Rightarrow 2^m t^m \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{d^m}{dx^m} H_n$$

$$= 2^m \sum_{n=0}^{\infty} \frac{H_n}{n!} t^{n+m} \qquad n+m=r$$

$$= 2^m \sum_{r=m}^{\infty} \frac{t^r}{(r-m)!} H_{r-m}(x)$$

t^n 'nin katsayısı : $= 2^m \dfrac{1}{(n-m)!} H_{n-m}$

$$\Rightarrow \frac{d^m}{dx^m} H_n = \frac{2^m n!}{(n-m)!} H_{n-m} \qquad n>m$$

2°) $\displaystyle\int_{-\infty}^{+\infty} x e^{-x^2} H_n(x) H_m(x)\, dx$ integralini hesaplayınız.

$$x H_n = n H_{n-1} + \frac{1}{2} H_{n+1}$$

$$\Rightarrow \int_{-\infty}^{+\infty} x e^{-x^2} H_n H_m\, dx = \int_{-\infty}^{+\infty} e^{-x^2} \left\{ n H_{n-1} + \frac{1}{2} H_{n+1} \right\} H_m\, dx$$

$$= n \int_{-\infty}^{+\infty} e^{-x^2} H_{n-1} H_m\, dx + \frac{1}{2} \int_{-\infty}^{+\infty} e^{-x^2} H_{n+1} H_m\, dx$$

$$= n\, 2^{n-1} (n-1)! \sqrt{\pi}\, \delta_{n-1,m} + \frac{1}{2} 2^{n+1} (n+1)! \sqrt{\pi}\, \delta_{n+1,m}$$

$$= \sqrt{\pi}\, 2^{n-1} n!\, \delta_{n-1,m} + \sqrt{\pi}\, 2^{n} (n+1)!\, \delta_{n+1,m}$$

3°) $\displaystyle P_n(x) = \frac{2}{\sqrt{\pi}\, n!} \int_0^{\infty} t^n e^{-t^2} H_n(xt)\, dt$ olduğunu gösteriniz.

$$H_n(xt) = \sum_{r=0}^{[n/2]} (-1)^r \frac{n!}{r!\,(n-2r)!} (2xt)^{n-2r}$$

$$\Rightarrow \frac{2}{\sqrt{\pi}\, n!} \int_0^{\infty} t^n e^{-t^2} H_n(xt)\, dt = \frac{2}{\sqrt{\pi}\, n!} \int_0^{\infty} t^n e^{-t^2}$$

$$\times \sum_{r=0}^{[n/2]} (-1)^r \frac{n!}{r!\,(n-2r)!} 2^{n-2r} x^{n-2r} t^{n-2r}\, dt$$

$$\Gamma(x) = 2\int_0^{\infty} t^{2x-1} e^{-t^2}\, dt$$

Ör. 2.4

$$= \sum_{r=0}^{[n/2]} \frac{2^{n-2r+1} (-1)^r x^{n-2r}}{\sqrt{\pi}\, r!\, (n-2r)!} \cdot \frac{1}{2} \cdot \Gamma\left(n-r+\frac{1}{2}\right)$$

$$= \sum_{r=0}^{[n/2]} \frac{2^{n-2r}\,(-1)^r\,x^{n-2r}}{\sqrt{\pi}\, r!\,(n-2r)!}\;\frac{(2n-2r)!}{2^{2n-2r}\,(n-r)!}\,\sqrt{\pi}$$

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\,\Gamma(x)\,\Gamma\left(x+\frac{1}{2}\right)$$

$$= \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{2^n\, r!\,(n-2r)!\,(n-r)!}\, x^{n-2r}$$

$$= P_n(x)$$

PROBLEMLER :

Prb. 1)

$$\int_{-\infty}^{+\infty} x^2 e^{-x^2}\{H_n\}^2\,dx = \sqrt{\pi}\; 2^n\, n!\;\left(n+\frac{1}{2}\right)$$

$$x H_n = n H_{n-1} + \frac{1}{2} H_{n+1}$$

$$\Rightarrow\; x^2 H_n^2 = n^2 H_{n-1}^2 + \frac{1}{4} H_{n+1}^2 + \cancel{2}\frac{n}{\cancel{2}} H_{n-1} H_{n+1}$$

$$\int_{-\infty}^{+\infty} e^{-x^2} H_n H_m\,dx = \sqrt{\pi}\; 2^n\, n!\;\delta_{nm}$$

$$\Rightarrow \int_{-\infty}^{+\infty} x^2 e^{-x^2} H_n^2\,dx = n^2 \int_{-\infty}^{+\infty} e^{-x^2} H_{n-1}^2\,dx$$

$$+ \frac{1}{4}\int_{-\infty}^{+\infty} H_{n+1}^2\, e^{-x^2}\,dx$$

$$+ \cancel{2}\frac{n}{\cancel{2}} \underbrace{\int_{-\infty}^{+\infty} e^{-x^2} H_{n-1} H_{n+1}}_{\to 0}$$

$$= n^2 \sqrt{\pi}\, 2^{n-1}(n-1)! + \frac{1}{4}\sqrt{\pi}\, 2^{n+1}(n+1)!$$

$$= n\sqrt{\pi}\, 2^{n-1} n! + \sqrt{\pi}\, 2^{n-1}(n+1)\, n!$$

$$= \sqrt{\pi}\, 2^{n-1} n!\;(n+n+1) = \sqrt{\pi}\; 2^n\, n!\;\left(n+\frac{1}{2}\right)$$

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2°) $f(x)$ m. mertebeden bir polinom ile

$$f(x)=\sum_{r=0}^{m} c_r H_r(x)$$

formunda yazılabilir ve burada

$$c_r=\frac{1}{2^r r!\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-x^2} f(x) H_r(x)\,dx$$

olduğunu gösteriniz.

$$f(x)=\sum_{r=0}^{m} c_r H_r(x)$$

$$\Rightarrow \int_{-\infty}^{+\infty} e^{-x^2} f(x) H_m(x)\,dx=\sum_{r=0}^{\infty} c_r \underbrace{\int_{-\infty}^{+\infty} e^{-x^2} H_r H_m\,dx}_{\sqrt{\pi}\,2^r r!\,\delta_{rm}}$$

$$=\sqrt{\pi}\,2^m m!\,c_m$$

$$\Rightarrow c_m=\frac{1}{2^m m!\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-x^2} f(x) H_m(x)\,dx.$$

$f(x)$'in derecesi m'den küçükse 0 //.

3°) $e^{-x^2} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} \cos 2xt \, dt$ olduğunu gösteriniz.

bu sonucu x'e göre $(2n)$ defa türeterek

$$H_{2n}(x) = \frac{2^{2n+1} (-1)^n e^{x^2}}{\sqrt{\pi}} \int_0^\infty e^{-t^2} t^{2n} \cos 2xt \, dt$$

olduğunu gösteriniz. (H_{2n+1} için de bir ifade bulunuz.)

$$H_n(x) = \frac{2^n (-i)^n e^{x^2}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2+2itx} t^n \, dt$$

olduğunu araştırınız.

<u>Çözüm</u>:

$$\int_0^\infty e^{-t^2} \cos 2xt = \frac{1}{\cancel{2}} \frac{\cancel{1}}{2} \int_{-\infty}^\infty e^{-t^2} \left(e^{2ixt} + e^{-2ixt} \right) dt$$

$$= \frac{1}{\cancel{2}2} \left\{ \int_{-\infty}^\infty e^{-t^2+2ixt} dt + \int_{-\infty}^\infty e^{-t^2-2ixt} dt \right\}$$

$$= \frac{1}{\cancel{2}2} \left\{ e^{-x^2} \int_{-\infty}^\infty e^{-(t-ix)^2} dt + e^{-x^2} \int_{-\infty}^{+\infty} e^{-(t+ix)^2} dt \right.$$

$$= \frac{\cancel{2}}{\cancel{4}_2} e^{-x^2} \int_{-\infty}^{+\infty} e^{-t^2} dt \; \sqrt{\pi}$$

$$\Rightarrow \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} \cos 2xt \, dt = e^{-x^2}$$

$$\frac{d}{dx}\cos 2xt = -2t\sin 2xt \qquad \frac{d^4}{dx^4}\cos 2xt = 2^4 t^4 \cos 2xt$$

$$\frac{d^2}{dx^2}\cos 2xt = -2^2 t^2 \cos 2xt$$

$$\frac{d^3}{dx^3}\cos 2xt = +2^3 t^3 \sin 2xt$$

$$\Rightarrow \begin{cases} \dfrac{d^{2n}}{dx^{2n}}\cos 2xt = (-1)^n 2^{2n} t^{2n} \cos 2xt \\[2ex] \dfrac{d^{2n+1}}{dx^{2n+1}}\cos 2xt = (-1)^{n+1} 2^{2n+1} t^{2n+1} \sin 2xt. \end{cases}$$

$$\cancel{\frac{d^{2n}}{dx^{2n}} e^{-x^2} = (-x^2)^{2n} e^{-x^2}}$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \quad , \text{idi}$$

$$H_{2n}(x) = (-1)^{2n} e^{x^2} \frac{d^{2n}}{dx^{2n}} e^{-x^2}$$

$$= (-1)^{2n} e^{x^2} \cdot \frac{d^{2n}}{dx^{2n}} \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} \cos 2xt \, dt$$

$$= \overset{1}{\cancel{(-1)^{2n}}} e^{x^2} \frac{2}{\sqrt{\pi}} \cdot (-1)^n 2^{2n} \int_0^\infty e^{-t^2} t^{2n} \cos 2xt \, dt$$

$$= \frac{2^{2n+1} (-1)^n e^{x^2}}{\sqrt{\pi}} \int_0^\infty e^{-t^2} t^{2n} \cos 2xt \, dt.$$

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$$H_{2n+1}(x) = (-1)^{2n+1} e^{x^2} \frac{d^{2n+1}}{dx^{2n+1}} e^{-x^2}$$

$$= - e^{x^2} \frac{d^{2n+1}}{dx^{2n+1}} \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} \cos 2xt \, dt$$

$$= - e^{x^2} \frac{2}{\sqrt{\pi}} \cdot (-1)^{n+1} 2^{2n+1} \int_0^\infty e^{-t^2} t^{2n+1} \sin 2xt \, dt$$

$$= \frac{(-1)^n 2^{2n+1} e^{x^2}}{\sqrt{\pi}} \int_0^\infty e^{-t^2} t^{2n+1} \sin 2xt \, dt.$$

$$n \to \frac{n}{2}$$

$$H_n(x) = \frac{2^{n+1} i^n e^{x^2}}{\sqrt{\pi}} \int_0^\infty e^{-t^2} t^n \cos 2xt \, dt.$$

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5.7. Weber – Hermite Fonksiyonları

$$\frac{d^2 y}{dx^2} + (\lambda - x^2)\, y = 0$$

$$y = z\, e^{-x^2/2}$$

$$\frac{d^2 z}{dx^2} - 2x \frac{dz}{dx} + (\lambda - 1)\, z = 0$$

$2n = \lambda - 1$ ile Hermite denklemi.

$$\Phi_n(x) = e^{-x^2/2} H_n(x) \qquad n = \frac{\lambda - 1}{2}$$

Weber fonk. ları.

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