# BÖLÜM 6

## LAGUERRE POLİNOMLARI

### 6.1. Laguerre Denk. i ve çözümü

$$x y'' + (1-x) y' + n y = 0$$

genellikle; $x$ 'in tüm sonlu değerleri için çözümün sonlu kalması, sonsuza giderken de $e^{x/2}$ 'den daha hızlı sonsuza gitmemesi istenir.

$$a_{r+1} = a_r \frac{(s+r-n)}{(s+r+1)^2}$$

$s=0$ 'da ikili kök. $\left\{ Z(x,0) \, , \left(\frac{\partial Z}{\partial s}\right)\Big|_{s=0} \right.$

2. 'si $\ln x$ ihtiva ederki bu $x=0$ 'da sonsuzdur.

$$a_{r+1} = a_r \frac{r-n}{(r+1)^2} \quad ; \quad s=0$$

Bu bağıntıdan elde edilecek seri $e^x$ gibi davranır bu ise yukarıda bahsettiğimiz sebeplerden dolayı $x$ 'in büyük değerleri için yeterince iyi davranışlı değildir. Bu durumda seriyi sonlandırırız. Bu da $n$ pozitif tamsayı alarak yapılabilir.

$$a_n \neq 0 \quad , \quad a_{n+1} = 0$$

$$a_{r+1} = -a_r \frac{n-r}{(r+1)^2}$$

$r \to r-1$

$$a_r = -a_{r-1} \frac{n-r+1}{(r)^2}$$

$$\left\{ \begin{array}{l} a_1 = -a_0 \dfrac{n}{1^2} \\[2ex] a_2 = -a_1 \dfrac{n-1}{2^2} = a_0 \dfrac{n(n-1)}{(2!)^2} \\ \vdots \end{array} \right.$$

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$$y = a_0 \left\{ 1 - \frac{n}{1^2} x + \frac{n(n-1)}{(2!)^2} x^2 + \cdots \right.$$

$$\left. (-1)^r \frac{n(n-1)\cdots(n-r+1)}{(r!)^2} x^r + \cdots \right\}$$

$$= a_0 \sum_{r=0}^{n} (-1)^r \frac{n(n-1)\cdots(n-r+1)}{(r!)^2} x^r$$

$$= a_0 \sum_{0}^{n} (-1)^r \frac{n!}{(n-r)!\,(r!)^2} x^r$$

$$a_0 = 1 \quad \Rightarrow \quad y \to L_n(x)$$

$$L_n(x) = \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!\,(r!)^2} x^r$$

6.2. Doğurucu Fonksiyon <u>thr. 6.1</u>

$$\frac{e^{-xt/(1-t)}}{1-t} = \sum_{0}^{\infty} L_n(x)\, t^n$$

<u>İspat:</u>

$$\frac{1}{1-t} e^{-xt/(1-t)} = \frac{1}{1-t} \sum_{0}^{\infty} \frac{1}{r!} \left( -\frac{xt}{1-t} \right)^r$$

$$= \sum_{0}^{\infty} \frac{(-1)^r}{r!} \frac{x^r t^r}{(1-t)^{r+1}}$$

$$\frac{1}{(1-t)^{r+1}} = 1 + (r+1)t + \frac{(r+1)(r+2)}{2!} t^2 + \cdots = \sum_{s=0}^{\infty} \frac{(r+s)!}{r!\,s!} t^s$$



<u>Binom thr:</u>

$$\Rightarrow \frac{1}{(1-t)} e^{-xt/(1-t)} = \sum_{r,s=0}^{\infty} (-1)^r \frac{(r+s)!}{(r!)^2 (s!)} x^r t^{r+s}$$

$$r+s=n \Rightarrow s=n-r$$

$t^n$:'nin katsayısı $\displaystyle\sum_{r=0}^{n} (-1)^r \frac{n!}{(r!)^2 (n-r)!} x^r = L_n$

6.3. Laguerre polinomları için diğer ifadeler:

Thr. 6.2

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} \left( x^n e^{-x} \right)$$

İspat: Leibniz thr: $\displaystyle\frac{d^n}{dx^n}(uv) = \sum_{r=0}^{n} \frac{n!}{(n-r)!} \frac{1}{r!} \frac{d^{n-r}}{dx^{n-r}} u \frac{d^r v}{dx^r}$

$$\Rightarrow \frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x}) = \frac{e^x}{n!} \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \frac{d^{n-r}}{dx^{n-r}} x^n \frac{d^r e^{-x}}{dx^r}$$

$$\frac{d^p}{dx^p} x^q = q(q-1) \dots (q-p+1) x^{q-p} = \frac{q!}{(q-p)!} x^{q-p}$$ $q=n$, $p=n-r$

$$\Rightarrow \frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x}) = \frac{\cancel{e^x}}{\cancel{n!}} \sum_{r=0}^{n} \frac{\cancel{n!}}{r!(n-r)!} \frac{n!}{r!} x^r (-1)^r \cancel{e^{-x}}$$

$$= \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!(r!)^2} x^r = L_n(x)$$

6.4 Laguerre polinomları'nın özel değerleri:

$$L_0(x) = 1 \quad , \quad L_1(x) = -x+1 \quad , \quad L_2(x) = \frac{1}{2!}\left(x^2-4x+2\right)$$

$$L_3(x) = \frac{1}{3!}\left(-x^3+9x^2-18x+6\right)$$

Thr. 6.3 i) $L_n(0) = 1$ , ii) $L_n'(0) = -n$

İspat. i) Doğurucu fonk.'da $x=0$

$$\frac{1}{1-t} = \sum_0^{\infty} L_n(0)\, t^n = \sum_0^{\infty} t^n$$

ii) $x L_n'' + (1-x) L_n' + n L_n = 0$

$$\Rightarrow L_n'(0) + n L_n(0) = 0 \Rightarrow L_n'(0) = -n$$

6.5. Laguerre Polinomlarının Ortogonalliği:

Thr. 6.4.

$$\int_0^{\infty} e^{-x} L_n(x) L_m(x)\, dx = \delta_{nm}$$

İspat:

$$\frac{e^{-xt/(1-t)}}{1-t} = \sum_{n=0}^{\infty} t^n L_n(x)$$

$$\frac{e^{-xs/(1-s)}}{1-s} = \sum_{m=0}^{\infty} s^m L_m(x)$$

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$$\sum_{n,m=0}^{\infty} e^{-x} L_n(x) L_m(x)\, t^n s^m = e^{-x} \frac{e^{-xt/(1-t)}}{1-t} \cdot \frac{e^{-xs/(1-s)}}{1-s}$$

$$I = \int_0^{\infty} e^{-x} \frac{e^{-xt/(1-t)}}{1-t} \cdot \frac{e^{-xs/(1-s)}}{1-s}\, dx$$

$$= \frac{1}{(1-t)(1-s)} \int_0^{\infty} \exp\left\{-x\left[1+\frac{t}{1-t}+\frac{s}{1-s}\right]\right\} dx$$

$$= \frac{1}{(1-t)(1-s)} \left[-\frac{1}{1+(t/(1-t))+(s/(1-s))} \exp\left\{-x\left[1+\frac{t}{1-t}+\frac{s}{1-s}\right]\right\}\right]_0^{\infty}$$

$$= \frac{1}{(1-t)(1-s)\left\{1+\frac{t}{1-t}+\frac{s}{1-s}\right\}} = \frac{1}{(1-t)(1-s)+t(1-s)+s(1-t)}$$

$$= \frac{1}{(1-t)[1-\cancel{s}+\cancel{s}]+t(1-s)} = \frac{1}{1-\cancel{t}+\cancel{t}-st} = \frac{1}{1-st} = \sum_{n=0}^{\infty} s^n t^n$$

$$\int_0^{\infty} e^{-x} L_n L_m\, dx = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases} = \delta_{nm}.$$

## 6.6. Rekürans Bağıntıları:

Th. 6.5.

(i) $(n+1) L_{n+1} = (2n+1-x) L_n - n L_{n-1}$

(ii) $x L_n' = n L_n - n L_{n-1}$ ; (iii) $L_n' = -\sum_{r=0}^{n-1} L_r(x)$

İspat:

(i) Doğuran fonk.'un $t$'ye göre türetilir.

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$$\sum_{n=0}^{\infty} L_n\, n\, t^{n-1} = \frac{1}{(1-t)^2} e^{-xt/(1-t)} - \frac{x}{(1-t)^2} \cdot \frac{e^{-xt/(1-t)}}{(1-t)}$$

$$= \frac{1}{1-t} \sum_{n=0}^{\infty} L_n t^n - \frac{x}{(1-t)^2} \sum_{n=0}^{\infty} L_n t^n$$

$$\Rightarrow (1-t)^2 \sum_{n=0}^{\infty} L_n\, n\, t^{n-1} = (1-t) \sum_{n=0}^{\infty} L_n t^n - x \sum_{n=0}^{\infty} L_n t^n$$

$$\downarrow \rightarrow \sum_{n=1}^{\infty} \rightarrow (n \rightarrow n+1)$$

$$\cancel{\sum_{n=0}^{\infty} L_n (n+1) t^n}$$

$$(1-2t+t^2) \sum_{n=0}^{\infty} (n+1) L_{n+1} t^n = (1-t) \sum_{n=0}^{\infty} L_n t^n - x \sum_{n=0}^{\infty} L_n t^n$$

$$\sum_{n=0}^{\infty} (n+1) L_{n+1} t^n - 2 \sum_{n=0}^{\infty} (n+1) L_{n+1} t^{n+1} + \sum_{n=0}^{\infty} (n+1) L_{n+1} t^{n+2}$$

$$= \sum_{n=0}^{\infty} L_n t^n - \sum_{n=0}^{\infty} L_n t^{n+1} - x \sum_{n=0}^{\infty} L_n t^n$$

$$\sum_{n=0}^{\infty} (n+1) L_{n+1} t^n - 2 \sum_{n=1}^{\infty} n L_n t^n + \sum_{n=2}^{\infty} (n-1) L_{n-1} t^n$$

$$= \sum_{n=0}^{\infty} L_n t^n - \sum_{n=1}^{\infty} L_{n-1} t^n - x \sum_{n=0}^{\infty} L_n t^n$$

$$\sum_{n=0}^{\infty} (n+1) L_{n+1} t^n - 2 \sum_{n=0}^{\infty} n L_n t^n + \sum_{n=1}^{\infty} (n-1) L_{n-1} t^n$$

$$= \sum_{n=0}^{\infty} L_n t^n - \sum_{n=1}^{\infty} L_{n-1} t^n - x \sum_{n=0}^{\infty} L_n t^n \qquad \sum_{n=0}^{\infty}$$

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$$(n+1)L_{n+1} - 2nL_n + (n-1)L_{n-1}$$

$$= L_n' - L_{n-1}' - x L_n'$$

$$(n+1)L_{n+1} = (2n+1-x)L_n - nL_{n-1}$$

(ii) Doğurucu fonk.'u $x$'e göre türet.

6.7. Birleşik Laguerre Polinomları:

$$x y'' + (k+1-x)y' + ny = 0$$

Thr. 6.6: Eğer $z$, $(n+k)$. mertebeden Laguerre denk.'inin bir çözümü ise $d^k z / dx^k$ Birleşik Laguerre denk.'ini sağlar.

$$z = L_{n+k} \qquad y$$

$$L_n^k = (-1)^k \frac{d^k}{dx^k} L_{n+k}(x)$$

bir koyuyorum.

Thr. 6.7.

$$L_n^k(x) = \sum_{r=0}^{n} (-1)^r \frac{(n+k)!}{(n-r)!\,(k+r)!\,r!} x^r$$

İspat: $L_{n+k}(x)$'i oluştur. daha sonra $x$'e göre $k$ kere türet.

## 6.8. Birleşik Laguerre polinomlarının özellikleri

Thr. 6.8. (Doğurucu Fonk.)

$$\frac{e^{-xt/(1-t)}}{(1-t)^{k+1}} = \sum_{n=0}^{\infty} L_n^k(x)\, t^n$$

İspat: normal Laguerre polinomları için doğurucu fonk.'u x'e göre k kere türet.

Thr. 6.9:

$$L_n^k = \frac{e^x x^{-k}}{n!} \frac{d^n}{dx^n}\left(e^{-x} x^{n+k}\right)$$

Thr. 6.10: (Ortogonallik)

$$\int_0^{\infty} e^{-x} x^k L_n^k L_m^k = \frac{(n+k)!}{n!}\delta_{nm}$$

Thr. 6.11: Rekürans bağıntıları

(i) $L_{n-1}^k + L_n^{k-1} = L_n^k$

(ii) $(n+1) L_{n+1}^k = (2n+k+1-x) L_n^k - (n+k) L_{n-1}^k$

(iii) $x L_n^{k\prime} = n L_n^k - (n+k) L_{n-1}^k$

(iv) $L_n^{k\prime} = -\sum_{r=0}^{n-1} L_r^k$

(v) $L_n^{k\prime} = -L_{n-1}^{k+1}$

(vi) $L_n^{k+1} = \sum_{r=0}^{n} L_r^k$

6.7. Birleşik Laguerre Polinomları:

$$x y'' + (k+1-x) y' + n y = 0$$

thr. 6.6. $z$, $(n+k)$. mertebeden Laguerre denk.'ni. bir çözümü ise $d^k z / dx^k$ birleşik L. Df. denk.'ini. çözümüdür.

İspat:

$$z = L_{n+k}$$

$$x z'' + (1-x) z' + (n+k) z = 0 \quad \text{denk.'nin çözümü.}$$

bu denk.'i $\underline{\underline{k}}$ kere türetir ve Leibniz formülünü kullanarak,

$$x \frac{d^{k+2}}{dx^{k+2}} z + k \frac{d^{k+1}}{dx^{k+1}} z + (1-x) \frac{d^{k+1}}{dx^{k+1}} z + k \cdot -1 \cdot \frac{d^k}{dx^k} z$$

$$+ (n+k) \frac{d^k}{dx^k} z = 0$$

$$\Rightarrow \quad x \frac{d^2}{dx^2} \left( \frac{d^k z}{dx^k} \right) + (k+1-x) \frac{d}{dx} \left( \frac{d^k z}{dx^k} \right) + n \frac{d^k z}{dx^k} = 0$$

$$L_n^k (x) = (-1)^k \frac{d^k}{dx^k} L_{n+k}(x)$$

Thr. 6.7. $$L_n^k = \sum_{r=0}^{n} (-1)^r \frac{(n+k)!}{(n-r)!\,(k+r)!\,r!} x^r$$

ispat :

$$L_{n+k} = \sum_{r=0}^{n+k} (-1)^r \frac{(n+k)!}{(n+k-r)!\,(r!)^2} x^r$$

$$L_n^k = (-1)^k \frac{d^k}{dx^k} \sum_{r=0}^{n+k} (-1)^r \frac{(n+k)!}{(n+k-r)!\,(r!)^2} x^r$$

$r < k \Rightarrow 0$

$$= (-1)^k \frac{d^k}{dx^k} \sum_{r=k}^{n+k} (-1)^r \frac{(n+k)!}{(n+k-r)!\,(r!)^2} x^r$$

$$\frac{d^k x^r}{dx^k} = r(r-1)\ldots(r-k+1)\, x^{r-k} = \frac{r!}{(r-k)!} x^{r-k}$$

$$= (-1)^k \sum_{r=k}^{n+k} (-1)^r \frac{(n+k)!}{(n+k-r)!\,(r!)^2} \frac{r!}{(r-k)!} x^{r-k}$$

$s = r-k$

$$= (-1)^k \sum_{s=0}^{n} (-1)^{k+s} \frac{(n+k)!}{(n-s)!\,(k+s)!\,s!} x^s$$

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6.8. Birleşik Laguerre Polinomlarının Özellikleri:

Thr. 6.8. (Doğurucu Fonk.)

$$\frac{e^{-xt/(1-t)}}{(1-t)^{k+1}} = \sum_{n=0}^{\infty} L_n^k(x)\, t^n$$

<u>ispat</u>:

$$\frac{e^{-xt/(1-t)}}{1-t} = \sum_{n=0}^{\infty} L_n(x)\, t^n$$

$x$'e göre $k$ kere türet.

Thr. 6.9.

$$L_n^k(x) = \frac{e^x x^{-k}}{n!} \frac{d^n}{dx^n}\left\{e^{-x} x^{n+k}\right\}$$

<u>ispat</u>: daha evvelki gibi! <u>Thr 6.2</u>

Thr. 6.10. (Ortogonallik Özelliği)

$$\int_0^{\infty} e^{-x} x^k L_n^k L_m^k \, dx = \frac{(n+k)!}{n!} \delta_{nm}$$

<u>ispat</u>: <u>Thr. 6.4.</u> 'ünkü gibi

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Thr. 6.11. (İndirgeme Bağıntıları)

(i) $L_{n-1}^{k} + L_{n}^{k-1} = L_{n}^{k}$

(ii) $(n+1)\, L_{n+1}^{k} = (2n+1+k-x)\, L_{n}^{k} - (n+k)\, L_{n-1}^{k}$

(iii) $x\, L_{n}^{k\,\prime} = n L_{n}^{k} - (n+k)\, L_{n-1}^{k}$

(iv) $L_{n}^{k\,\prime} = -\sum\limits_{r=0}^{n-1} L_{r}^{k}$

(v) $L_{n}^{k\,\prime} = -L_{n-1}^{k+1}$ (vi). $L_{n}^{k+1} = \sum\limits_{n=0}^{n} L_{r}^{k}$

İspat:

$$L_{n-1}^{k} + L_{n}^{k-1} = \sum_{r=0}^{n-1} (-1)^r \frac{(n-1+k)!}{(n-1-r)!\,(k+r)!\,r!} x^r + \sum_{r=0}^{n} (-1)^r \frac{(n+k-1)!}{(n-r)!\,(k-1+r)!\,r!} x^r$$

$$= \sum_{r=0}^{n-1} (-1)^r \frac{(n+k-1)!}{(n-r-1)!\,(k+r)!\,r!} x^r + \sum_{r=0}^{n-1} (-1)^r \frac{(n+k-1)!}{(n-r)!\,(k+r-1)!\,r!} x^r$$

$$+ (-1)^n \frac{(n+k-1)!}{(n-n)!\,(k-1+n)!\,n!} x^n$$

$$= \sum_{r=0}^{n-1} (-1)^r \frac{(n+k-1)!}{(n-r-1)!\,(k+r-1)!\,r!} \left\{ \frac{1}{k+r} + \frac{1}{n-r} \right\} x^r + (-1)^n \frac{x^n}{n!}$$

$$= \sum_{r=0}^{n-1} (-1)^r \frac{(n+k-1)!}{(n-r-1)!\,(k+r-1)!\,r!} \, \frac{n-\cancel{r}+k+\cancel{r}}{(k+r)(n-r)} x^r + (-1)^n \frac{x^n}{n!}$$

$$= \sum_{r=0}^{n-1} (-1)^r \frac{(n+k)!}{(n-r)!\,(k+r)!\,r!} x^r + (-1)^n \frac{x^n}{n!}$$

$$= \sum_{r=0}^{n} (-1)^r \frac{(n+k)!}{(n-r)!\,(k+r)!\,r!} x^r = L_{n}^{k}$$

(iv) thm. 6.5 (iii) $\quad L_n' = -\sum_{r=0}^{n-1} L_r$

bunu k kere türetip $\quad n \to n+k$

$$\underbrace{\frac{d^k}{dx^k} L'_{n+k}} = -\sum_{r=0}^{n+k-1} \frac{d^k}{dx^k} L_r \qquad L_n^k = (-1)^k \frac{d^k}{dx^k} L_{n+k}$$

$$(-1)^k L_n^{k\prime} = -\sum_{r=k}^{n+k-1} \frac{d^k}{dx^k} L_r$$

$r<k \to 0$

$$(-1)^k L_n^{k\prime} = -\sum_{s=0}^{n-1} \frac{d^k}{dx^k} L_{s+k} \qquad s = r-k$$

$$= -\sum_{s=0}^{n-1} (-1)^k L_s^k$$

$$L_n^{k\prime} = -\sum_{r=0}^{n-1} L_r^k$$

6.10. Örnekler:

Örnek 1. $L_n^{\alpha+\beta+1}(x+y) = \sum_{r=0}^{n} L_r^{\alpha}(x) L_{n-r}^{\beta}(y)$

olduğunu ispatlayınız.

Thr. 6.8.) $e^{-xt/(1-t)}/(1-t)^{k+1} = \sum_{n=0}^{\infty} L_n^{k} t^n$

$$\Rightarrow \sum_{n=0}^{\infty} \underbrace{L_n^{\alpha+\beta+1}(x+y)}_{t^n\text{'nin katsayısı.}} t^n = \frac{e^{-(x+y)t/(1-t)}}{(1-t)^{\alpha+\beta+2}}$$

$$\frac{e^{-(x+y)t/(1-t)}}{(1-t)^{\alpha+\beta+2}} = \frac{e^{-xt/(1-t)}}{(1-t)^{\alpha+1}} \frac{e^{-yt/(1-t)}}{(1-t)^{\beta+1}}$$

$$= \sum_{r=0}^{\infty} L_r^{\alpha}(x) t^r \sum_{s=0}^{\infty} L_s^{\beta}(y) t^s$$

$$= \sum_{r,s=0}^{\infty} L_r^{\alpha}(x) L_s^{\beta}(y) t^{r+s}$$

$r+s=n$

$$\Rightarrow \quad t^n: \sum_{r=0}^{n} L_r^{\alpha}(x) L_{n-r}^{\beta}(y)$$

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Örnek 2.

$$J_m\{2\sqrt{xt}\} = e^{-t}(xt)^{m/2}\sum_{n=0}^{\infty}\frac{L_n^m(x)}{(n+m)!}t^n$$

$$e^{t}(xt)^{-m/2}J_m\{2\sqrt{xt}\} = \sum_{n=0}^{\infty}\frac{L_n^m}{(n+m)!}t^n$$

$$J_m\{2\sqrt{xt}\} = \sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,(m+r)!}\left\{\frac{\cancel{2}\sqrt{xt}}{\cancel{2}}\right\}^{2r+m}$$

$$e^{t}(xt)^{-m/2}J_m\{2\sqrt{xt}\} = e^{t}\cancel{(xt)^{-m/2}}\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!}\frac{1}{(m+r)!}(xt)^{r+\cancel{(m/2)}}$$

$$= e^{t}\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,(m+r)!}(xt)^r = \sum_{s=0}^{\infty}\frac{t^s}{s!}\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,(m+r)!}(xt)^r$$

$$= \sum_{r,s}(-1)^r\frac{1}{r!\,(m+r)!\,s!}x^r t^{r+s}$$

$t^n$'nin katsayısı: $L_n^m/(n+m)!$ / $r+s=n$, $s=n-r$

$$(-1)^r\frac{1}{r!\,(m+r)!\,(n-r)!}x^r$$

$t^n$'nin katsayısı: $= \sum_{r=0}^{n}(-1)^r\frac{1}{r!\,(m+r)!\,(n-r)!}x^r$

$$= \frac{1}{(n+m)!}\sum_{r=0}^{n}(-1)^r\frac{(n+m)!}{r!\,(m+r)!\,(n-r)!}x^r = \frac{L_n^m}{(n+m)!}$$

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Örnek 3. $I = \int_x^\infty e^{-t} \underbrace{L_n^k}_{u} dt = e^{-x}\{ L_n^k - L_{n-1}^k \}$

kısmi integrasyon ile

$$I = -e^{-t} L_n^k(t) \Big|_x^\infty - \int_x^\infty -e^{-t} L_n^{k\prime} dt$$

$$= e^{-x} L_n^k(x) + \int_x^\infty e^{-t} L_n^{k\prime} dt$$

$$= e^{-x} L_n^k(x) - \int_x^\infty e^{-t} \left\{ \sum_{r=0}^{n-1} L_r^k(t) \right\} dt \quad (\text{Thr. } 6.11.\text{iv.})$$

$$\Rightarrow \sum_{r=0}^{n-1} \int_x^\infty e^{-t} L_r^k(t) dt + \int_x^\infty e^{-t} L_n^k(t) dt = e^{-x} L_n^k(x)$$

$$\Rightarrow \sum_{r=0}^{n} \int_x^\infty e^{-t} L_r^k(t) dt = e^{-x} L_n^k(x)$$

$$\int_x^\infty e^{-t} L_n^k(t) dt = \sum_{r=0}^{n} \int_x^\infty e^{-t} L_r^k(t) dt - \sum_{r=0}^{n-1} \int_x^\infty e^{-t} L_r^k(t) dt$$

$$= e^{-x} L_n^k(x) - e^{-x} L_{n-1}^k(x)$$

$$= e^{-x} \left[ L_n^k - L_{n-1}^k \right]$$

PROBLEMLER :

prb.1.) $L_n''(0) = \frac{1}{2} n(n-1)$ olduğunu gösteriniz.

$$x L_n'' + (1-x) L_n' + n L_n = 0$$

$$L_n'' + x L_n''' - L_n' + (1-x) L_n'' + n L_n' = 0$$

$$L_n''(0) - L_n'(0) + L_n''(0) + n L_n'(0) = 0$$

$$2 L_n''(0) = -(n-1) \underbrace{L_n'(0)}_{=-n}$$

$$L_n''(0) = \frac{1}{2} n(n-1)$$

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prb.2.) $f(x)$ m. mertebeden bir polinom ise, $f(x)$

$$f(x) = \sum_{r=0}^{m} C_r L_r(x) \qquad C_r = \int_0^\infty e^{-x} L_r(x)\, dx$$

şeklinde seriye açılabilir.

$$\int_0^\infty e^{-x} L_n(x)\, x^k = \begin{cases} 0 & k < n \end{cases}$$

$$f(x) = \sum_{r=0}^{m} C_r L_r(x)$$

$$\Rightarrow \int_0^\infty f(x) e^{-x} L_{r'}(x)\, dx = \sum_{r=0}^{m} C_r \underbrace{\int_0^\infty e^{-x} L_r L_{r'}\, dx}_{\delta_{rr'}}$$

$$= C_{r'}$$

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$$f(x) = x^k$$

$$\Rightarrow \int_0^\infty e^{-x} x^k L_n(x)\, dx = \int_0^\infty \cancel{e^{-x}} . x^k . \frac{\cancel{e^{x}}}{n!} \frac{d^n}{dx^n}\left(x^n e^{-x}\right) dx$$

$$= \int_0^\infty dx\, x^k \frac{d^n}{dx^n}\left(x^n e^{-x}\right) = \int_0^\infty \underbrace{x^k}_{u} \frac{d}{dx} \underbrace{\frac{d^{n-1}}{dx^{n-1}}\left(x^n e^{-x}\right)}_{dv} dx$$

$$= x^k \frac{d^{n-1}}{dx^{n-1}}\left(x^n e^{-x}\right)\Big|_0^\infty - k \int_0^\infty x^{k-1} \frac{d^{n-1}}{dx^{n-1}}\left(x^n e^{-x}\right) dx$$

$$= (-1)^n k(k-1) \dots (k-n) \int_0^\infty x^{k-n} x^n e^{-x}\, dx \qquad k < n \Rightarrow 0$$

$$= (-1)^n \frac{k!}{(k-n-1)!} \underbrace{\int_0^\infty x^k e^{-x}\, dx}_{\Gamma(k+1) = k!} = (-1)^n \frac{(k!)^2}{(k-n-1)!}$$

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$$I = \int_0^\infty e^{-x} x^{k+1} {L_n^k}^2 \, dx$$

$$\int_0^\infty e^{-x} x^{k} {L_n^k}^2 \, dx = \frac{(n+k)!}{n!}$$

$$\ni \int_0^\infty e^{-x} x^{k+1} {L_n^{k+1}}^2 \, dx = \frac{(n+k+1)!}{n!} = (n+k+1)\frac{(n+k)!}{n!}$$

$$L_n^{k+1} = L_n^k + L_{n-1}^{k+1} \qquad , \; -L_{n-1}^{k+1} = {L_n^k}'$$

$$I2 = \underbrace{\int_0^\infty e^{-x} x^{k+1} {L_n^k}^2 \, dx} + \int_0^\infty e^{-x} x^{k+1} {L_{n-1}^{k+1}}^2 \, dx = \frac{(n+k)!}{(n-1)!}$$

$$+ 2 \int_0^\infty e^{-x} L_n^k L_{n-1}^{k+1} x^{k+1} \, dx = (n+k+1)\frac{(n+k)!}{n!}$$

$$\therefore I = (n+k+1)\frac{(n+k)!}{n!} - \frac{(n+k)!}{(n-1)!} + 2\int_0^\infty e^{-x} x^{k+1} L_n^k {L_n^k}' \, dx$$

$$x {L_n^k}' = n L_n^k - (n+k) L_{n-1}^k$$

$$= (n+k+1)\frac{(n+k)!}{n!} - n\frac{(n+k)!}{n!} + 2\left\{ n\int_0^\infty e^{-x} x^k {L_n^k}^2 \, dx = \frac{(n+k)!}{n!} \right.$$

$$\left. - (n+k)\cancel{\int_0^\infty e^{-x} x^k L_n^k L_{n-1}^k \, dx} = 0 \right\}$$

$$= \frac{(n+k)!}{n!}\left[ (n+k+1) - n + 2n \right] \checkmark$$

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$$\int_{-1}^{+1} \{P_\ell^m(x)\}^2 dx = \int_{-1}^{+1} (1-x^2)^m \left[\frac{d^m}{dx^m} P_\ell\right]\left[\frac{d^m}{dx^m} P_\ell\right]$$

$$= \int_{-1}^{+1} \underbrace{(1-x^2)^m \frac{d^m P_\ell}{dx^m}}_{u} \underbrace{\frac{d}{dx}\left[\frac{d^{m-1}}{dx^{m-1}} P_\ell\right] dx}_{dv}$$

$$= uv - \int_{-1}^{+1} \frac{d^{m-1}}{dx^{m-1}} P_\ell \frac{d}{dx}\left[(1-x^2)^m \frac{d^m P_\ell}{dx^m}\right] dx.$$

$$(1-x^2)\frac{d^2 z_1}{dx^2} - 2(m+1)x\frac{dz_1}{dx} + [\ell(\ell+1) - m(m+1)] z_1 = 0$$

$$z_1 = \frac{d^m z}{dx^m} \qquad m \to m-1$$

$$(1-x^2) P_\ell^{(m+1)} - 2mx P_\ell^{(m)} + [\ell(\ell+1) - m(m-1)] P_\ell^{(m-1)} = 0$$

$$(1-x^2)^{m-1} \text{ile çarp.} \qquad \begin{array}{c} \ell^2 - m^2 + \ell + m \\ (\ell+m)(\ell-m+1) \end{array}$$

$$\underbrace{(1-x^2)^m P_\ell^{(m+1)} - 2mx P_\ell^{(m)} (1-x^2)^{m-1}}_{} P_\ell^{(m)} + (\ell+m)(\ell-m+1) P_\ell^{(m-1)} (1-x^2)^{m-1} = 0.$$

$$\frac{d}{dx}\left[(1-x^2)^m \frac{d^m}{dx^m} P_\ell\right] = -(\ell+m)(\ell-m+1)\,(1-x^2)^{m-1} P_\ell^{(m-1)}$$

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$$\frac{1}{r} = p^2 q + b \cos\frac{\theta}{p} = p^2 q \left[1 + \frac{b}{p^2 q} \cos\frac{\theta}{p}\right]$$

$$r = \frac{1/p^2 q}{1 + \varepsilon \cos\frac{\theta}{p}} = \frac{1/p^2 q}{1 + \varepsilon \cos \alpha\theta}$$

$$\boxed{r = \frac{a(1-\varepsilon^2)}{1 + \varepsilon \cos \alpha\theta}}$$

BS K

$$\int_{-1}^{+1} \left[ P_\ell^m(x) \right]^2 dx = (\ell+m)(\ell-m+1) \int_{-1}^{+1} (1-x^2)^{m-1} \left[ \frac{d^{m-1}}{dx^{m-1}} P_\ell \right] \left[ \frac{d^{m-1}}{dx^{m-1}} P_\ell \right]$$

$$= (\ell+m)(\ell-m+1) \int_{-1}^{+1} \left[ P_\ell^{m-1}(x) \right]^2 dx$$

$$\cancel{= (\ell+m)(\ell-m+1)}$$

$$= (\ell+m)(\ell-m+1)(\ell+m-1)(\ell-m+2) \int_{-1}^{+1} \left[ P_\ell^{m-2}(x) \right]^2 dx.$$

$$= (\ell+m)(\ell+m-1) \cdots (\ell+1)(\ell-m+1)(\ell-m+2) \cdots (\ell$$

$$= (\ell+m)(\ell+m-1) \cdots (\ell+1)\ell(\ell-1) \cdots (\ell-m+2)(\ell-m+1) \frac{2}{2\ell+1}$$

$$= \frac{(\ell+m)!}{(\ell-m)!} \frac{2}{2\ell+1}$$

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$m<0$ olsun. $n>0$ $\Rightarrow$ $m=-n$

$$\int_{-1}^{+1} [P_\ell^m(x)]^2 dx = \int_{-1}^{+1} [P_\ell^{-n}(x)]^2$$

$$= \left[ (-1)^n \frac{(\ell-n)!}{(\ell+n)!} \right]^2 \int_{-1}^{+1} [P_\ell^n(x)]^2 dx .$$

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$$\boxed{(2l+1)\,x\,P_l^m = (l+m)\,P_{l-1}^m + (l-m+1)\,P_{l+1}^m}$$

$$(l+1)\,P_{l+1} - (2l+1)\,x\,P_l + l\,P_{l-1} = 0$$

$$\cancel{(l+1)\,P_{l+1}^{(m)} - (2l+1)\,x\,P_l^{(m)} + l\,P_{l-1}^{(m)} - (2l+1)}$$

$$(l+1)\,P_{l+1}^{(m)} - (2l+1)\left[\,x\,P_l^{(m)} + m\,P_l^{(m-1)}\right] + l\,P_{l-1}^{(m)} = 0$$

$$P_{l+1}^{(1)} - P_{l-1}^{(1)} = (2l+1)\,P_l \qquad (m-1).$$

$$P_{l+1}^{(m)} - P_{l-1}^{(m)} = (2l+1)\,P_l^{(m-1)}$$

$$(l+1)\,P_{l+1}^{(m)} - (2l+1)\,x\,P_l^{(m)} - m\left[P_{l+1}^{(m)} - P_{l-1}^{(m)}\right] = -l\,P_{l-1}^{(m)}$$

$$(1-x^2)^{m/2}$$

BS K

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$$\int_{-1}^{+1} \{P_\ell^m(x)\}^2 dx = \int_{-1}^{+1} (1-x^2)^m \left[\frac{d^m}{dx^m}P_\ell\right]\left[\frac{d^m}{dx^m}P_\ell\right]$$

$$= \int_{-1}^{+1} \underbrace{(1-x^2)^m \frac{d^m P_\ell}{dx^m}}_{u} \underbrace{\frac{d}{dx}\left[\frac{d^{m-1}}{dx^{m-1}}P_\ell\right]dx}_{dv}$$

$$= uv - \int_{-1}^{+1} \frac{d^{m-1}}{dx^{m-1}}P_\ell \,\frac{d}{dx}\left[(1-x^2)^m \frac{d^m P_\ell}{dx^m}\right]dx.$$

$$(1-x^2)\frac{d^2 z_1}{dx^2} - 2(m+1)x\frac{dz_1}{dx} + [\ell(\ell+1)-m(m+1)]z_1 = 0$$

$$z_1 = \frac{d^m z}{dx^m} \qquad m \to m-1$$

$$(1-x^2)P_\ell^{(m+1)} - 2mxP_\ell^{(m)} + [\ell(\ell+1)-m(m-1)]P_\ell^{(m-1)} = 0$$

$(1-x^2)^{m-1}$ ile çarp. $\qquad \ell^2-m^2+\ell+m$

$\qquad (\ell+m)(\ell-m+1)$

$$(1-x^2)^m P_\ell^{(m+1)} - 2mxP_\ell^{(m)}(1-x^2)^{m-1} + (\ell+m)(\ell-m+1)P_\ell^{(m-1)}(1-x^2)^{m-1} = 0.$$

$$\frac{d}{dx}\left[(1-x^2)^m\frac{d^m}{dx^m}P_\ell\right] = -(\ell+m)(\ell-m+1)(1-x^2)^{m-1}P_\ell^{(m-1)}$$

BS K

$$\frac{1}{r} = p^2 q + b \cos\frac{\theta}{p} = p^2 q \left[ 1 + \frac{b}{p^2 q} \cos\frac{\theta}{p} \right]$$

$$r = \frac{1/p^2 q}{1 + \varepsilon \cos\frac{\theta}{p}} = \frac{1/p^2 q}{1 + \varepsilon \cos \alpha\theta}$$

$$\boxed{r = \frac{a(1-\varepsilon^2)}{1 + \varepsilon \cos \alpha\theta}}$$

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$$\int_{-1}^{+1} \left[ P_\ell^m(x) \right]^2 dx = (\ell+m)(\ell-m+1) \int_{-1}^{+1} (1-x^2)^{m-1} \left[ \frac{d^{m-1}}{dx^{m-1}} P_\ell \right] \left[ \frac{d^{m-1}}{dx^{m-1}} P_\ell \right]$$

$$= (\ell+m)(\ell-m+1) \int_{-1}^{+1} \left[ P_\ell^{m-1}(x) \right]^2 dx$$

$$\cancel{= (\ell+m)(\ell-m+1)}$$

$$= (\ell+m)(\ell-m+1)(\ell+m-1)(\ell-m+2) \int_{-1}^{+1} \left[ P_\ell^{m-2}(x) \right]^2 dx.$$

$$= (\ell+m)(\ell+m-1) \cdots (\ell+1)(\ell-m+1)(\ell-m+2) \cdots (\ell$$

$$= (\ell+m)(\ell+m-1) \cdots (\ell+1)\ell(\ell-1) \cdots (\ell-m+2)(\ell-m+1) \frac{2}{2\ell+1}$$

$$= \frac{(\ell+m)!}{(\ell-m)!} \frac{2}{2\ell+1}$$

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$m<0$ olsun. $n>0 \Rightarrow m=-n$

$$\int_{-1}^{+1} [P_\ell^m(x)]^2 dx = \int_{-1}^{+1} [P_\ell^{-n}(x)]^2$$

$$= \left[ (-1)^n \frac{(\ell-n)!}{(\ell+n)!} \right]^2 \int_{-1}^{+1} [P_\ell^n(x)]^2 dx.$$

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$$\boxed{(2\ell+1)\,x\,P_\ell^m = (\ell+m)\,P_{\ell-1}^m + (\ell-m+1)\,P_{\ell+1}^m}$$

$$(\ell+1)\,P_{\ell+1} - (2\ell+1)\,x\,P_\ell + \ell\,P_{\ell-1} = 0$$

$$\cancel{(\ell+1)\,P_{\ell+1}^{(m)} - (2\ell+1)\,x\,P_\ell^{(m)} + \ell\,P_{\ell-1}^{(m)} - (2\ell+1)}$$

$$(\ell+1)\,P_{\ell+1}^{(m)} - (2\ell+1)\left[\,x\,P_\ell^{(m)} + m\,P_\ell^{(m-1)}\right] + \ell\,P_{\ell-1}^{(m)} = 0$$

$$P_{\ell+1}^{(1)} - P_{\ell-1}^{(1)} = (2\ell+1)\,P_\ell \qquad (m-1).$$

$$P_{\ell+1}^{(m)} - P_{\ell-1}^{(m)} = (2\ell+1)\,P_\ell^{(m-1)}$$

$$(\ell+1)\,P_{\ell+1}^{(m)} - (2\ell+1)\,x\,P_\ell^{(m)} - m\left[P_{\ell+1}^{(m)} - P_{\ell-1}^{(m)}\right] = -\ell\,P_{\ell-1}^{(m)}$$

$$(1-x^2)^{m/2}$$

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