BÖLÜM 12

BOSE SİSTEMLERİ

12.1. Fotonlar

Elektromag. alan → herbiri farklı frekans ile salınan n salınıcı

$$\left(n+\frac{1}{2}\right)\hbar\omega \qquad n=0,1,2,3,\dots$$

Fotonlar EMA'nın kuantumu.

↳ Q. Teorisine göre $\hbar$ spinli kütlesiz bozonlar.

↳ kütlesizlik fotonların c hızı ile "serbest uzayda" hareket ettiğini gösterir.

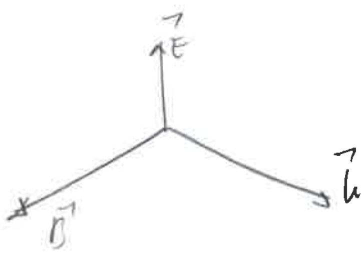
↳ spini sadece iki bağımsız yönelim olasılığı; paralel anti paralel $\uparrow\uparrow$, $\downarrow\uparrow$ ($\vec{p}$ mom. a)

Foton'un belirli bir spin durumunda bulunması ya sağ elli ya da sol elli çembersel polarize bir düzlem EMA karşılık gelir. Bununla birlikte, belirli spinli iki foton durumunu süperimpoze edebiliriz ve spinin özdurumu olmayan ancak lineer polarize olan bir foton durumu elde edebiliriz. Aşağıda lineer polarize fotonları alacağız.

Enerji $= E = \hbar\omega$

Momentum $= \vec{p} = \hbar\vec{k}$ $\qquad |\vec{k}| = \dfrac{\omega}{c}$

pol. vektörü $= \vec{\epsilon}$ $\qquad |\vec{\epsilon}| = 1$ $\qquad \vec{k}\cdot\vec{\epsilon} = 0$



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Bir kutuda, $\vec{k} = \frac{2\pi}{L}\vec{n}$ $\quad n = 0, \pm 1, \pm 2, \pm 3, \dots$

izin verilmiş mom. durumlarının sayısı $k - k + dk$

$$\frac{V}{(2\pi)^3} 4\pi k^2 \, dk$$

atomlar foton yayınlayıp soğurabildiklerinden toplam foton sayısı korunmaz.

$$E\{n_{\vec{k},\epsilon}\} = \sum_{\vec{k},\vec{\epsilon}} \hbar\omega \, n_{\vec{k},\epsilon} \qquad \omega = c|\vec{k}|$$

$$n_{\vec{k},\vec{\epsilon}} = 0, 1, 2, \dots$$

Sayısı korunmadığından

$$Q = \sum_{\{n_{\vec{k},\vec{\epsilon}}\}} e^{-\beta E\{n_{\vec{k},\vec{\epsilon}}\}} = \prod_{\vec{k},\vec{\epsilon}} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} = \prod_{\vec{k},\vec{\epsilon}} \frac{1}{1 - e^{-\beta\hbar\omega}}$$

$$Q = \prod_{\vec{k},\vec{\epsilon}} \frac{1}{1 - e^{-\beta\hbar\omega}}$$

$$\ln Q = -\sum_{\vec{k},\vec{\epsilon}} \ln\left(1 - e^{-\beta\hbar\omega}\right) = -2\sum_{\vec{k}} \ln\left(1 - e^{-\beta\hbar\omega}\right)$$

$$\langle n_{\vec{k}} \rangle = -\frac{1}{\beta}\frac{\partial}{\partial(\hbar\omega)} \ln Q = \frac{2}{e^{\beta\hbar\omega} - 1}$$

$$U = -\frac{\partial}{\partial\beta} \ln Q = \sum \hbar\omega_k \langle n_k \rangle$$

$$P = \frac{1}{\beta}\frac{\partial}{\partial V} \ln Q \qquad \ln Q = -2\sum_{k} \ln\left(1 - e^{-\beta\hbar c \frac{2\pi}{V^{1/3}} |\vec{n}|}\right)$$

$$PV = \frac{1}{3} U$$

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$V \to \infty$ limitinde, $\sum_k \to \frac{V}{8\pi^3}\int d^3\vec{k}$

$$\frac{U}{V} = \frac{\hbar}{\pi^2 c^3}\int_0^\infty d\omega\, \frac{\omega^3}{e^{\beta\hbar\omega}-1} \qquad U = \frac{2V}{8\pi^3}\int_0^\infty dk\, 4\pi k^2\, \frac{\hbar c k}{e^{\beta\hbar c k}-1}$$

$$= \int_0^\infty d\omega\, u(\omega,T) \qquad u(\omega,T) = \frac{\hbar}{\pi^2 c^3}\,\frac{\omega^3}{e^{\beta\hbar\omega}-1}$$

Planck ışıma yasası.

$$\int_0^\infty dx\, \frac{x^3}{e^x-1} = \frac{\pi^4}{15}$$

$$\frac{U}{V} = \frac{\pi^2}{15}\,\frac{(kT)^4}{(\hbar c)^3}$$

$$C_V = \frac{1}{V}\left(\frac{\partial U}{\partial T}\right)_V = \frac{4\pi^2 k^4 T^3}{15(\hbar c)^3}$$

$T \to \infty$ giderken bounded değil. Kavitedeki foton sayısı bounded olmadığından

böyle bir kavite açılarak deneysel olarak bu yasa test edilebilir. Birim alan başına birim zamanda ışıma miktarı;

$$I(\omega,T) = \frac{c}{2}\int \frac{d\Omega}{4\pi}\, u(\omega,T)\cos\theta = \frac{c}{4}\, u(\omega,T)$$

$$I(T) = \alpha T^4$$

$T_1 < T_2 < \ldots$

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12.2. Fononlar:

$\mathcal{H} = T + V$

N atom var $\Rightarrow$ $3N$ normal mod. $\omega_1, \omega_2, \dots, \omega_{3N}$

$$E\{n_i\} = \sum_{i=1}^{3N} n_i \hbar\omega_i \qquad Q = \sum_{\{n_i\}} e^{-\beta E\{n_i\}} = \prod_{i=1}^{3N} \frac{1}{1 - e^{-\beta\hbar\omega_i}}$$

$$\ln Q = -\sum_{i=1}^{3N} \ln\left(1 - e^{-\beta\hbar\omega_i}\right)$$

$$\langle n_i \rangle = -\frac{1}{\beta}\frac{\partial}{\partial(\hbar\omega_i)} \ln Q = \frac{1}{e^{\beta\hbar\omega_i} - 1}$$

$$U = -\partial_\beta \ln Q = \sum_{i=1}^{3N} \hbar\omega_i \langle n_i \rangle = \sum \frac{\hbar\omega_i}{e^{\beta\hbar\omega_i} - 1}$$

$V \to \infty$ iken $\quad \dfrac{U}{V} = \dfrac{3}{2\pi^2 c^3} \displaystyle\int_0^{\omega_m} d\omega\, \omega^2 \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}$

ω — $\omega + d\omega$ arasındaki normal kip'lerin sayısı $= f(\omega)\,d\omega = 3\,\dfrac{V}{8\pi^3}\,4\pi k^2\,dk$

mümkün polarizasyonlar

$$= V\,\frac{3\omega^2}{2\pi^2 c^3}\,d\omega$$

$$\int_0^{\omega_m} f(\omega)\,d\omega = 3N \qquad v = V/N \qquad \omega_m = c\left(\frac{6\pi^2}{v}\right)^{1/3}$$

$$\lambda_m = \frac{2\pi c}{\omega_m} = \left(\frac{4}{3}\pi v\right)^{1/3}$$

parçacıklar arası mesafe.

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$$\frac{U}{N} = \frac{9(kT)^4}{(\hbar\omega_m)^3} \int_0^{\beta\hbar\omega_m} dt \, \frac{t^3}{e^t - 1}$$

$$D(x) \equiv \frac{3}{x^3} \int_0^x dt \, \frac{t^3}{e^t - 1} = \begin{cases} 1 - \frac{3}{8}x + \frac{1}{20}x^2 + \cdots & x \ll 1 \\ \frac{\pi^4}{5x^3} + O(e^{-x}) & x \gg 1 \end{cases}$$

Debye func.

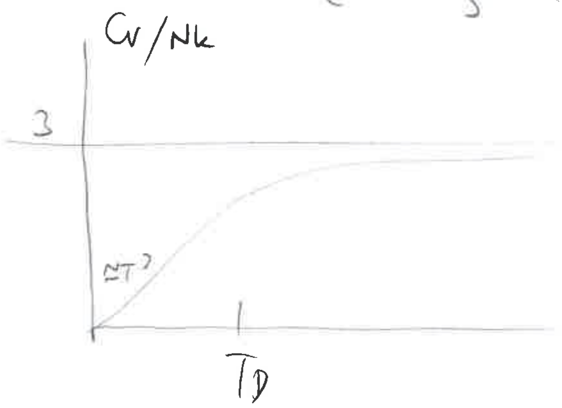
$$T_D \equiv \hbar\omega_m = \hbar c \left(\frac{6\pi^2}{v}\right)^{1/3}$$

$$\therefore \frac{U}{N} = 3kT\,D(x) = \begin{cases} 3kT\left(1 - \frac{3T_D}{8T} + \cdots\right) & T \gg T_D \\ 3kT\left(\frac{\pi^4}{5}\left(\frac{T}{T_D}\right)^3 + \cdots\right. & T \ll T_D \end{cases}$$

$x = T_D/T$

$$\frac{C_V}{Nk} = 3D(x) + 3T\frac{dD(x)}{dT} = 3\left(4D(x) - \frac{3x}{e^x - 1}\right]$$

$$= \begin{cases} 3\left[1 - \frac{1}{20}\left(\frac{T_D}{T}\right)^2 + \cdots\right] & T \gg T_D \\ \frac{12\pi^4}{5}\left(\frac{T}{T_D}\right)^3 + \cdots & T \ll T_D \end{cases}$$



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12.3. Bose-Einstein Yoğuşması :

$$\frac{1}{v} = \frac{1}{\lambda^3} g_{3/2}(z) + \frac{1}{V}\frac{z}{1-z} \qquad v = \frac{V}{N} \qquad \lambda = \sqrt{2\pi\hbar^2/mkT}$$

(i incelemeliyiz)

$$0 \leq z \leq 1$$

$$G_n(z) = \int_0^\infty \frac{x^{n-1}}{z^{-1}e^x - 1}\,dx$$

$$\lim_{z\to 0} G_n(z) = \int_0^\infty z\, x^{n-1} e^{-x}\,dx = \Gamma(n)$$

$$G_n(z) = \Gamma(n)\, g_n(z) \Rightarrow g_n(z) = \frac{1}{\Gamma(n)} G_n(z)$$

$$g_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{z^{-1}e^x - 1}\,dx$$

küçük z'le için

$$g_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{z^{-1}e^x(1-ze^{-x})}\,dx$$

$$= \frac{1}{\Gamma(n)} \int_0^\infty x^{n-1} z e^{-x}\left(1 + ze^{-x} + z^2e^{-2x} + \dots\right)dx$$

$$= \frac{1}{\Gamma(n)} \int_0^\infty x^{n-1} \sum_{\ell=1}^{\infty} (ze^{-x})^\ell$$

$$= \frac{1}{\Gamma(n)} \sum_{\ell=1}^{\infty} z^\ell \int x^{n-1} e^{-x\ell}\,dx \qquad x\ell = u$$

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$$= \frac{1}{\Gamma(n)} \sum_{l=1}^{\infty} z^l \int \frac{u^{n-1}}{l^{n-1}} e^{-u} \frac{du}{l}$$

$$= \frac{1}{\cancel{\Gamma(n)}} \sum z^l \frac{\cancel{\Gamma(n)}}{l^n} = \sum_{l=1}^{\infty} \frac{z^l}{l^n} = z + \frac{z^2}{2^n} + \frac{z^3}{3^n} + \cdots$$

$$G_n(1) = \sum_{l=1}^{\infty} \frac{1}{l^n} = \zeta(n) \qquad n > 1$$

$n < 1$ için ıraksar.

$$G_2(1) = \zeta(2) = \frac{\pi^2}{6} \simeq 1.645$$

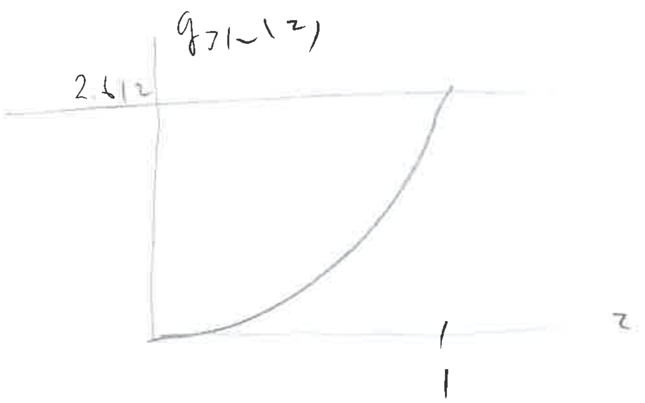
$$G_4(1) = \zeta(4) = \frac{\pi^4}{90} \simeq 1.082$$

$$\zeta(6) = \frac{\pi^6}{945} \simeq 1.017$$

$$\zeta(8) = \frac{\pi^8}{9450} \simeq 1.004$$

$$G_{3/2}(1) = \zeta(3/2) \simeq 2.612$$

$$\frac{z}{1-z} = \langle n_0 \rangle \qquad \frac{\lambda^3 \langle n \rangle}{V} = \frac{\lambda^3}{v} - g_{3/2}(z) \qquad \frac{\lambda^3}{v} \geq g_{3/2}(z)$$



z'yi T'nin fonk. olarak çözmek için bunu grafiksel çözmek gerek.

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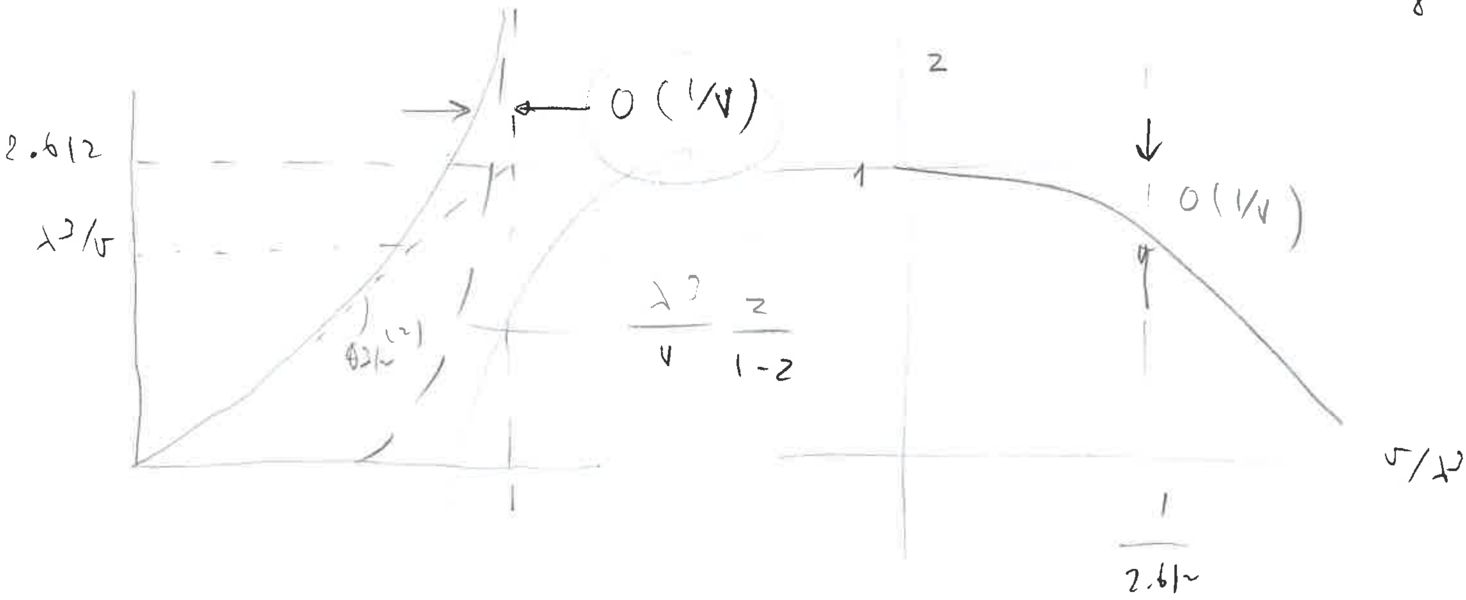

$V \to \infty$ iken 0 olur.

$$\frac{\lambda^3}{v} = g_{3/2}(1) \qquad (V, T) \to V_c,\ T_c$$

$$\left(\frac{2\pi\hbar^2}{mkT_c}\right)^{3/2} = g_{3/2}(1) \quad \text{hacim sbt.}$$

$$kT_c = \frac{2\pi\hbar^2/m}{\left[v\, g_{3/2}(1)\right]^{2/3}} \qquad v_c = \frac{\lambda^3}{g_{3/2}(1)}$$

$$\frac{\lambda^3}{v} \geq g_{3/2}(1) = 2.612 \qquad \frac{\lambda^3}{v} = \frac{\langle n_0\rangle \lambda^3}{V} + g_{3/2}(1) \quad z=1$$

BE yoğuşması

$$\langle n_0 \rangle = \frac{z}{1-z}$$

$$\frac{\lambda^3}{V}\langle n_0\rangle = \frac{N\lambda^3}{V} - g_{3/2}(1)$$

$$\frac{\langle n_0\rangle}{N} = 1 - \frac{V}{N}\frac{1}{\lambda^3} g_{3/2}(1) = 1 - \frac{g_{3/2}(1)}{g_{3/2}(z)}$$

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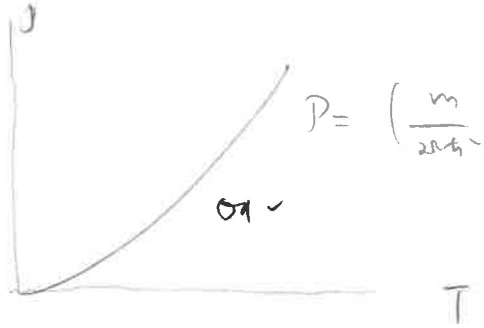
$$\frac{P}{kT} = \begin{cases} \frac{1}{\lambda^3} g_{5/2}(z) & v > v_c \\ \frac{1}{\lambda^3} g_{5/2}(1) & v < v_c \end{cases}$$

$v < v_c$ $\qquad P_0 = \frac{kT}{\lambda^3} g_{5/2}(1)$

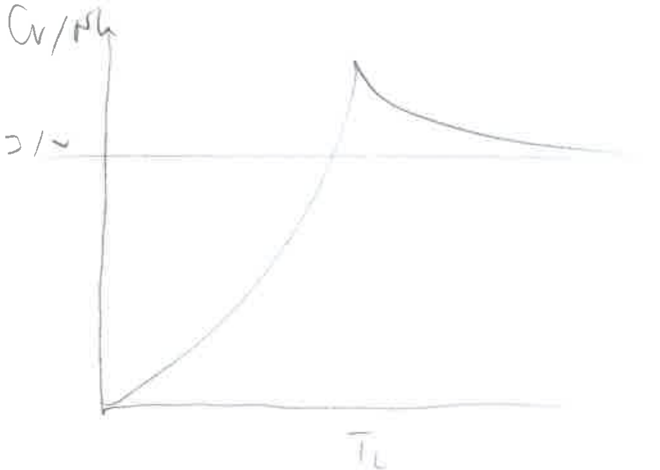
$$\frac{dP_0}{dT} = \frac{1}{T v_c}\left[\frac{5}{2} kT \frac{g_{5/2}(1)}{g_{3/2}(1)}\right]$$



$P = \left(\frac{m}{2\pi\hbar^2}\right)^{3/2} g_{3/2}(1)(kT)^{5/2}$

$$L = \frac{5}{2} kT \frac{g_{5/2}(1)}{g_{3/2}(1)}$$

$$\frac{C_V}{kT} = \begin{cases} (15/4)(v/\lambda^3)\, g_{5/2}(1) & T < T_c \\ (15/4)(v/\lambda^3)\, g_{5/2}(z) - (9/4)\left(g_{3/2}(z)/g_{1/2}(z)\right) & T > T_c \end{cases}$$



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